Complex Numbers

Introduction

If we try to solve $x^2 = -1$, what happens?
We extract square roots to get $x = +/- \sqrt{-1}$. But if we try to evaluate the square root of –1 on a scientific calculator, we get ERROR! But still, we need a way to define solutions like this so it is defined that

$\mathbf{i^2 = -1}$ **and thus** $\mathbf{i = \sqrt{(-1)}}$**.**

This means that the solutions of $x^2 = -1$ *are* $x = i$ *and* $x = -i$
We refer to such solutions as **Complex Solutions**.

Furthermore, we refer to a number containing the quantity "i", where $i = \sqrt{-1}$, as an *imaginary number*. This choice of words "imaginary" is actually not appropriate, since we use the number "i" in many *real-world* engineering applications!

Using Complex Numbers To Evaluate Square Roots

Given that $i^2 = -1$ and thus $i = \sqrt{(-1)}$, we can use this fact to evaluate any square root.
For example, $\sqrt{(-13)} = \sqrt{(-1 \bullet 13)} = \sqrt{(-1)} \bullet \sqrt{13}$ and we can replace $\sqrt{(-1)}$ with i to get $\sqrt{(-13)} = i \bullet \sqrt{13}$.

In general, we can say that
$\sqrt{-A} = i\sqrt{A}$ **or** $\sqrt{A}\ i$
where "-A" is a negative Real Number

This allows us to write the answer to any solution of a quadratic equation in terms of i - If not for "i", Electrical Engineers would not be able to solve differential equations associated with applications. Another real-life application requiring "i" is shown at
http://www.mathmotivation.com/science/complex-numbers-application.html

Standard Form of a Complex Number

Complex Numbers consist of the set of all numbers of the form a + bi where, "a" is the *Real Part* and "bi" is the *Imaginary Part*.

It turns out that *all numbers* may be written in this form. For numbers that are regular old Real numbers, there is no i-part so b=0. For example, we may write 6 as 6 + 0i. Some numbers, like **3** + **2**i, have both a real and imaginary part, with **a** = **3** and **b** = **2**. And some numbers, like 4i have no Real part and may be written as 0 + 4i. We sometimes call numbers like 4i, that have no Real part, as *purely imaginary*.

Operations With Complex Numbers

All the standard properties that apply to Real Numbers, like the Distributive, Commutative, and Associative Properties, *also* apply to Complex Numbers. This means that you can use the following simple rule:

EASY RULE OF THUMB
When performing operations with complex numbers, treat the "i" like any other variable, except replace any occurrence of i^2 with –1.

Example: Multiply out (3 + 2i)(5 – 6i) and write the answer in *Standard Form*.

We simply multiply this out using the Distributive Property (FOIL Method) *as if* we were multiplying out (3 + 2x)(5 – 6x).

(3 + 2i)(5 – 6i)
= $3 \bullet 5 - 3 \bullet 6i + 2i \bullet 5 - 2i \bullet 6i$ by the Distributive Property
= $15 - 18i + 10i - 12i^2$ after multiplying terms
= $15 - 8i - 12\mathbf{i^2}$ after combining like terms
= $15 - 8i - 12(\mathbf{-1})$ after replacing i^2 with –1
= 27 – 8i or 27 + (-8i) after combining like terms

The Complex Conjugate
For any complex number a + bi, the *complex conjugate* is defined as a – bi.
If we multiply (a+bi)(a – bi), we get

(a+bi)(a – bi)
= $a^2 - abi + abi - b^2i^2$
= $a^2 - b^2\mathbf{i^2}$
= $a^2 - b^2(\mathbf{-1})$
= $a^2 + b^2$
The *i-part* always disappears!

Example: Multiply out (3 + 2i)(3 – 2i)
(3 + 2i)(3 – 2i)
= $3 \bullet 3 - 3 \bullet 2i + 2i \bullet 3 - 2i \bullet 2i$ by the Distributive Property
= $9 - 6i + 6i - 4i^2$ after multiplying terms
= $9 - 4\mathbf{i^2}$ after combining like terms
= $9 - 4(\mathbf{-1})$ after replacing i^2 with –1
= 13 after combining like terms

Or, if you used the generalization, you end up with $a^2 + b^2 = 3^2 + 2^2 = 13$.

Dividing Complex Numbers
To divide by a complex number, we can always get an answer in complex standard form by using the following rule:

$$\frac{c + di}{a + bi} \bullet \frac{a - bi}{a - bi} = \frac{ca - cb \bullet i + ad \bullet i - db \bullet i^2}{a^2 + b^2} = \frac{(ca+db) + (ad - cb) \bullet i}{a^2 + b^2} = \frac{ca+db}{a^2 + b^2} + \frac{ad - cb}{a^2 + b^2} \bullet i$$

Multiplying the numerator and denominator by the complex conjugate of the denominator will cause the denomiator to turn into a real number - it always works!

Do not panic! You don't need to memorize this formula. Simply remember to multiply top and bottom of the fraction by the conjugate of the denominator. Simplify your answer and it will all work out. See the next page for an example.

Example: Divide (3 + 2i) ÷ (4 –3i) and write the answer in *standard form.*

First, write this as a fraction.

$$\frac{3+2i}{4-3i}$$

$$= \frac{3+2i}{4-3i} \cdot \frac{4+3i}{4+3i}$$ Here, you are multiplying by a fraction equal to 1.

$$= \frac{12+9i+8i+6i^2}{16+12i-12i-9i^2}$$ Apply the Distributive Property

$$= \frac{12+17i+6(-1)}{16-9(-1)}$$ Combine like terms, Replace i^2 with -1

$$= \frac{6+17i}{25}$$ Combine like terms

$$= \frac{6}{25}+\frac{17}{25}i$$ Apply Distributive Property to split up fraction