Variation and Mathematical Modeling

When we know the relationship between two or more variables, we can then write an equation relating the variables to each other. In the sciences and business there are many formulas that describe the variation between quantities relating to the application. We refer to these formulas as *mathematical models* of the application. In order to formulate the model, we need to understand *what type of variation* is occurring.

Direct Variation

When two variables, x & y vary directly, then **y = k•x** where **k is a constant** called the "constant of proportionality." When there is direct variation between variables x and y, doubling the value of x will always result in y being doubled. Also, if x is cut in half, then y is cut in half.

An Example of Direct Variation

If y represents the miles traveled by a person driving down the highway at a speed of 50 MPH and x represents the hours spent traveling, then y = 50x and x and y vary directly. We also say that x is directly proportional to y. So if the person travels for x = 2 hours, they will cover y = 50(2) = 100 miles. If the person travels for x = 4 hours, then they will cover y = 50(4) = 200 miles. A doubling of the hours x results in a doubling of the miles y.

Example: In chemistry, we learn that for an ideal gas, pressure P varies directly as temperature T if all other variables are held constant. What is the equation relating P to T?

Here, P is related to T in the same way x is related to y in the definition of direct variation. So we can write: **P = k•T** . Another acceptable answer would be **T = k•P**.

Example: If S varies directly as T, and S = 40 when T = 5, write an equation relating S to T. Use the given data to find the value of k.

Since S and T vary directly, S = k •T. Letting S = 40 and T = 5 results in 40 = k • 5 which results in k = 8. The completed model is S = 8T.

Note: We could have also used the model T = k•S. We could solve for k in the same way and get T = (1/8)S. If we were to use the Multiplication Property of Equality to multiply both sides of this equation by 8, we would end up with S = 8T.

Inverse Variation

When two variables, x & y vary inversely, then **y = k•(1/x)** *or* **y = k•(1/x)** *or* **x•y = k** where **k is a constant** called the "constant of proportionality." When there is inverse variation between variables x and y, doubling the value of x will always result in y being cut in half. Also, if x is cut in half, then y is doubled.

Example: Write An Equation Relating Pressure To Volume

In chemistry, we learn that for an ideal gas, pressure P varies inversely as volume V if all other variables are held constant. What is the equation relating P to V?

Here, P is related to V in the same way x is related to y in the definition of inverse variation. So we can write: **PV = k**. This is in fact known as Boyle's Law.

Example: If Q varies inversely as W, and Q = 200 when W = 5, write an equation relating Q to W. Use the given data to find the value of k.
Since Q and W vary inversely, $Q = k \bullet (1/W)$. Letting Q = 200 & W = 5 results in $200 = k(1/5)$, which results in k = 1000. The completed model is $Q = 1000(1/W)$ or $QW = 1000$.

Joint Variation

If z varies jointly as x and y, then **$z = k \bullet xy$** where **k is a constant** of proportionality. Variables that vary jointly are also known as being "jointly proportional" to each other.

Example: z varies jointly as the square of x and the cube of y and we know that z = 100 when x = 5 and y = 2. What is the model relating z to x and y? Use the given data to find the value of k.
Here z does not equal kxy. In this case, $z = kx^2 y^3$ since x is squared and y is cubed. Letting z=100, x=5, and y =2 results in $100 = k(5^2)(2^3)$ or 100 = 200k. Applying the Division Property of Equality and solving results in 100/200 = ½ = k.
So the *completed model* is **$z = \frac{1}{2} x^2y^3$**.

Combinations of Variation

Often, several different types of variation will be included in one relationship.

Example: Write An Equation Relating Pressure, Volume, and Temperature
In chemistry, we learn that for an ideal gas, pressure P varies inversely as volume V and directly as temperature T, if all other variables are held constant. What is the equation relating P, V, and T?

Here, since P is related inversely to V, we have $P = k \bullet (1/V)$. Also, we have P varying directly as T. We use the same constant for both variations and we write:
$P = k \bullet (1/V)$ $\bullet$ T or, using the Multiplication Property of Equality to multiply both sides by V, we get **PV = kT**. Since we do not have values for P,V, and T, we can not solve for k.

This now resembles the Ideal Gas Law, PV = nRT, where R is a constant, and n represents the amount of the gas. In this example, n is constant, so the quantity nR is equivalent to our constant k.

Applications of Variation
As you can see from the examples so far, there are many applications of variation in the world around us. For example, the speed at which an object falls, neglecting air resistance, is directly proportional to the time elapsed.

Procedure For Doing Variation Application Problems

1. **Translate the words into an equation** containing the variables representing given quantities and a constant k. This is the "incomplete" model.
2. **Substitute given values of the variables** into the incomplete model and solve for the value of k. **Write the "complete" model** with the known value of k.
3. You may **use the completed model to predict behavior of variables** within the completed model.

Example: It is known that the diameter of the largest particle that can be moved by a stream varies approximately as the square of the velocity of the stream. It is known that a stream with a velocity of 0.5 miles per hour can move coarse sand particles about 0.04 inch in diameter. Approximate the velocity required to carry particles 0.15 inch in diameter.

First, define variables. Then, write down the model relating the variables.
If D = diameter of particle in inches, and V = velocity of stream in MPH, then

$D = k V^2$ is the model relating D to V.

Now, use the data V=0.5, D=0.04 to find the value of k.
$0.04 = k(0.5^2)$
$0.04 = 0.25k$
$0.04/0.25 = k = 0.16$ by the Division Property of Equality
Substitute this k-value into the model to get
$D = 0.16V^2$ ←This is your completed model

Now you can use this completed model to find the velocity required to move a particle with diameter D = 0.15 inches. **Plug D=0.15 into $D = 0.16V^2$** to get
$0.15 = 0.16V^2$
Now, solve for V.

$0.15/0.16 = V^2$	Divide both sides by 0.16 using the Division Property of Equality
$\sqrt{(0.15/0.16)} = V$	Extract Square Roots – but only use the positive case here
$0.96824... = V$	Evaluate Square Root
1.0 MPH = V	Round answer

Notes:
- o The answer was rounded to the nearest tenth to match the precision of the given data 0.5MPH which was carried out to the tenths place.
- o Make sure all your data units are consistent! If our given diameter would have been given in feet, we would have had to convert it to inches since our completed model used D in inches.