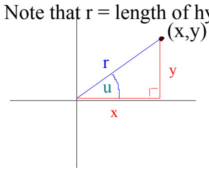
Wecome to Section 6.3 - Trigonometric Functions of Any Angle

Trigonometric Functions in the First Quadrant

If "u" is an angle in standard position with (x,y) being a point on the terminal side, then r = length of the segment connecting (0,0) to (x,y) as shown below.

Furthermore, by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$.
Note that r = length of hypotenuse.



For angles in the first quadrant, we may define sine, cosine, and tangent in the following way:

sin u = y/r,   cos u = x/r,   tan u = y/x
csc u = r/y,   sec u = r/x,   cot u = x/y

Trigonometric Functions in ANY Quadrant

We will use the same definitions of sine, cosine, tangent, etc. to determine the values of the trig functions at angles in quadrants II, III, & IV. We will also use what is called a "reference angle" and "reference triangle". SEE NEXT PAGE . . .

Trigonometric Functions in ANY Quadrant

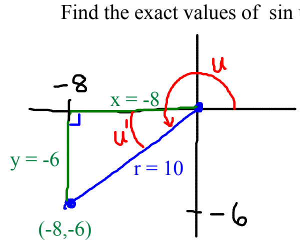
To find the trigonometric function of any angle u, you will do the following:

1. Construct a "reference triangle". This is a right triangle with hypotenuse equal to the terminal side of the angle, and the base equal to the x-axis. The acute angle formed by the x-axis and the terminal side is called the "reference angle".

2. Fill in the side lengths of the reference angle, using the correct sign on the x & y values.

3. Use the definitions sin u = y/r, cos u = x/r, tan u = y/x, csc u = r/y, sec u = r/x, cot u = x/y

Example: An angle u has a terminal side with terminal point (-8, -6) as shown below. Find the exact values of sin u, cos u, and tan u.



We construct a reference triangle that is a right triangle with the terminal side equal to the hypotenuse. We call the angle u' the reference angle. By the Pythagorean Theorem,

$$r = \sqrt{6^2 + 8^2} = 10$$

sin u = y/r = -6/10
cos u = x/r = -8/10
tan u = y/x = -6/(-8) = 3/4

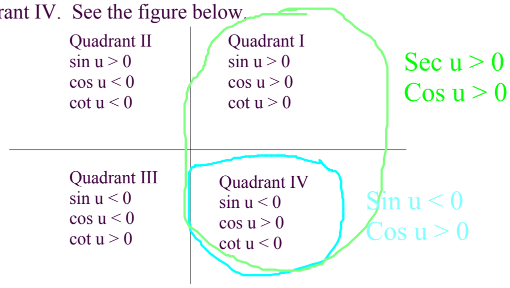
Notice that these values may be found by using SOHCAHTOA for the reference triangle.

Example: If sec u > 0 and cot u < 0, in what quadrant does "u" lie in?

If sec u > 0, cos u > 0 since sec u = 1/(cos u).

If cot u < 0, then either cos u < 0 and sin u > 0 OR cos u > 0 and sin u < 0 since cot u = (cos u)/(sin u). The latter case MUST be true because cos u > 0 by the first condition.

This means that cos u > 0 and sin u < 0. Cos u > 0 in quadrants I & IV. Sin u < 0 in quadrants III & IV. The common quadrant is IV so u is in quadrant IV. See the figure below.



Trigonometric Functions of Quadrant Angles

For the angles 90 deg, 180 deg, 270 deg, and 0 deg, and all coterminal angles, the trig functions are found with the same definitions as before where r = length of terminal side. We can not construct a reference triangle since the reference angle is either 0 degrees or 90 degrees. Note that "r" is always positive.

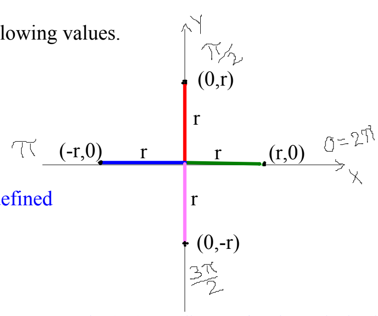
These quadrant angles have the following values.

sin 0 = y/r = 0/r = 0
cos 0 = x/r = r/r = 1
tan 0 = y/x = 0/1 = 0

sin 90 = sin pi/2 = y/r = r/r = 1
cos 90 = cos pi/2 = x/r = 0/r = 0
tan 90 = tan pi/2 = y/x = 1/0 = undefined

sin 180 = sin pi = y/r = 0/r = 0
cos 180 = cos pi = x/r = -r/r = -1
tan 180 = tan pi = y/x 0/-r = 0

sin 270 = 3pi/2 = y/r = -r/r = -1
cos 270 = 3pi/2 = x/r = 0/r = 0
tan 270 = 3pi/2 = y/x = -r/0 = undefined



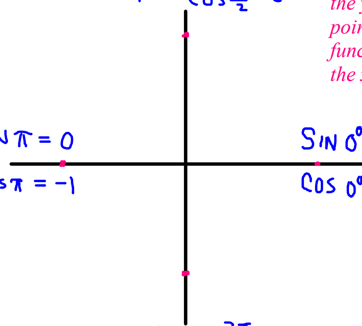
The (x,y) coordinates of each terminal point are given above. The x & y values are equal to zero or +r or -r.

It's probably easiest just to memorize the following diagram.

$\sin 90^\circ = \sin \frac{\pi}{2} = 1$
$\cos 90^\circ = \cos \frac{\pi}{2} = 0$

Remember that the sine function is determined by the y-value at the terminal point and the cosine function is determined by the x-value.

$\sin 180^\circ = \sin \pi = 0$
$\cos 180^\circ = \cos \pi = -1$



$\sin 0^\circ = \sin 0 = 0$
$\cos 0^\circ = \cos 0 = 1$

$\sin 270^\circ = \sin \frac{3\pi}{2} = -1$
$\cos 270^\circ = \cos \frac{3\pi}{2} = 0$

Example: Find the values of the following functions of quadrant angles:

$\sec \frac{5\pi}{2}$, $\cot 7\pi$

sec 5pi/2 = sec pi/2 since 5pi/2 = 2.5pi = 2pi + pi/2 .

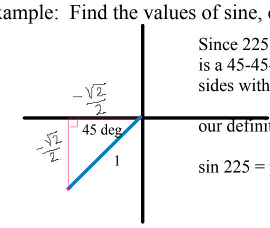
sec pi/2 = 1/cos pi/2 = 1/0 which is undefined.

Note that we found the functions of the coterminal angles.

cot 7pi = cot pi since 7pi = 6pi + pi.

cot pi = (cos pi)/(sin pi) = -1/0 which is also undefined.

Example: Find the values of sine, cosine, and tangent of 225 deg.



Since 225 is 45 more than 180, the reference triangle for this angle is a 45-45-90 triangle with reference angle of 45 degrees and sides with lengths labeled as (-sqrt 2)/2 and (-sqrt 2)/2. Using our definitions for sine, cosine, and tangent results in

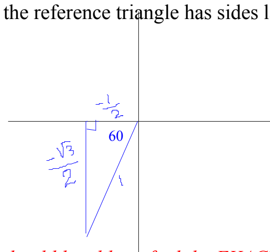
sin 225 = y/r = $-\frac{\sqrt{2}}{2}$ cos 225 = x/r = $-\frac{\sqrt{2}}{2}$ tan 225 = y/x = 1

Finding Trigonometric Functions of Angles with Special-Case Reference Angles

In the last example, the angle 225 degrees corresponded to a reference angle that was 45 degrees and we were able to find the exact value of the trigonometric functions. We may find the trig functions of ANY angle that is a multiple of 30, 60, or 45 degrees.

Example: Find sine, cosine, and tangent of $\frac{22\pi}{3}$.

Since 22pi/3 = 7pi + pi/3 = 6pi + 4pi/3, this angle is coterminal with 4pi/3. 4pi/3 is equal to 240 degrees = 180 degrees + 60 degrees, so the reference angle is 60 deg and the reference triangle has sides labeled -1/2, (-sqrt 3)/2, and 1.



Using the definitions for sine, cosine, and tangent results in

sin (22pi/3) = y/r = $\frac{\left(-\frac{\sqrt{3}}{2}\right)}{1} = -\frac{\sqrt{3}}{2}$

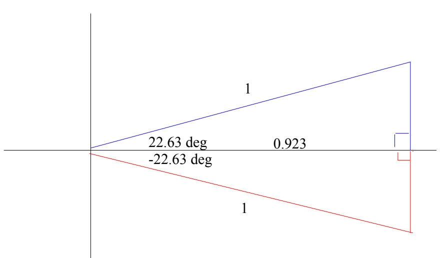
cos (22pi/3) = x/r = $\left(-\frac{1}{2}\right)/1 = -\frac{1}{2}$

tan (22pi/3) = y/x = $\left(\frac{-\sqrt{3}}{2}\right)/\left(-\frac{1}{2}\right) = \sqrt{3}$

You should be able to find the EXACT value of ANY angle corresponding to a special case reference triangle!

Example: Use your calculator to find TWO values of "u" where "u" is between 0 and 360 degrees where cos u = 0.923 . Round "u" to 2 decimal places.

We may find the first value easily by using the inverse cosine function. We get u = 22.63 degrees, rounded to 2 places. Now we must think "What other angle has this same cosine value?"



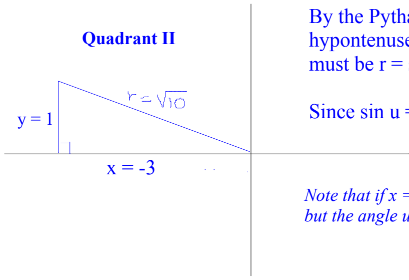
The reference triangle for 22.63 deg has sides labeled x = 0.923 and r = 1 where cos u = x/r.

Another reference triangle with the same side lengths is the one for -22.63 degrees which is equal to (360-22.63)= 337.37 degrees.

Our two angles are u = 22.63 degrees and u = 337.37 degrees

Example: Given that cot u = -3 and "u" is in quadrant II, find sin u.

Since cot u = x/y and angle "u" is in quadrant II, we must construct a reference triangle with sides of length x = -3 and y = 1.



By the Pythagorean Theorem, the hypontenuse of the reference triangle must be r = sqrt(10).

Since sin u = y/r, sin u = $\frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

Note that if x = 3 and y = -1, we get cot u = -3 but the angle u is in quadrant IV.

Do the exercises for Section 6.3 and read the text!