Welcome to Section 6.6 - Inverse Trigonometric Functions

The inverse of a function f(x) is defined as another function g(x) such that f [g(x)] = x and g[f(x)] = x.

What this means is that the inverse is another function such that if you place an x-value into f(x) and obtain a y-value, we may input this y-value into g(x) and the original x-value is obtained as a result.

For example, if f(x) = 3x + 2, its inverse is g(x) = (x - 2)/3.

If x = 5, f(5) = 3(5) + 2 = 17. We place 17 into g(x) to get (17 - 2)/3 = 5.

The inverse is a function that gives you the x-value that resulted in a given y-value.

For example, if f(x) = 10^x, Its inverse is g(x) = LOG x, where this is a base-10 log.

If we wish to find "What x-value results in 30 = 10^x?" , we place 30 into the LOG function to get LOG 30 = 1.47712 (rounded). Thus 10^1.47712 = 30.

In a similar way, we need functions that allow us to solve for x if y = sin x, y = cos x, and y = tan x.

Inverse Function of y = sin x

We may define the inverse of y = sin x as simply

y = arcsin x where sin y = x.

Thus, if y = 1 and y = sin x, then we may use the inverse function to find the angle x that results in 1 = sin x.

We plug x = 1 into y = arcsin x
to get y = arcsin 1.

We may rewrite this as sin y = 1. The angle we get is y = pi/2.

BUT WAIT!!!! Are there other values of y such that sin y = 1?

How about y = 5pi/2, y = -pi/2, y = 9pi/2, etc. which means we have numerous y-values obtained for the x-value of x = 1.

What is wrong with this function y = arcsin x? SEE NEXT PAGE . . .

y = arcsin x is NOT a function if the domain and range are not restricted because we get numerous y-values for any x-value we input.

We must RESTRICT the domain and range of y = arcsin x.

If y = arcsin x, we may always rewrite this as x = sin y. The possible x-values that we may obtain are given by the interval from -1 up to 1. We pick the smallest range of y-values that will give us all the possible x-values. We choose y to range from -pi/2 up to pi/2.

Thus, the inverse sine FUNCTION is given by

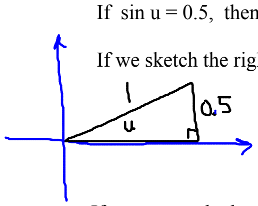
$$y = \arcsin x \quad \text{WHERE } -1 \le x \le 1 \quad \underline{\text{AND}} \ -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Note also that we may ALWAYS rewrite y = arcsin x as sin y = x.

Example: If sin u = 0.50, use the inverse function to find the value of "u". Find the value of "u" with AND without a calculator. SEE NEXT PAGE. . .

If sin u = 0.5, then arcsin (0.5) = u .

If we sketch the right triangle with sin u = 0.5/1, we obtain



We recognize this as a 30-60-90 triangle where u = 30 degrees or pi/6.

If we use a calculator, we enter 0.5 and we press the $\sin^{-1}$ key.

The answer you get will be 30 if you are in degrees mode or pi/6 = 0.523598 . . . if radians mode is used.

Definition of the Arccosine Function

For the same reasons as cited for the inverse sine function, we must restrict the domain and range of the inverse cosine function.

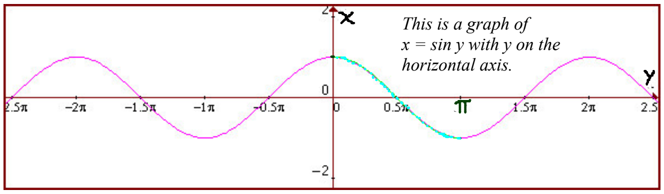
y = arccos x may be rewritten as cos y = x where $-1 \le x \le 1$, $0 \le y \le \pi$.

NOTE: We MUST restrict the domain and range!

Why are the y-values restricted the way they are here???? See Next Page

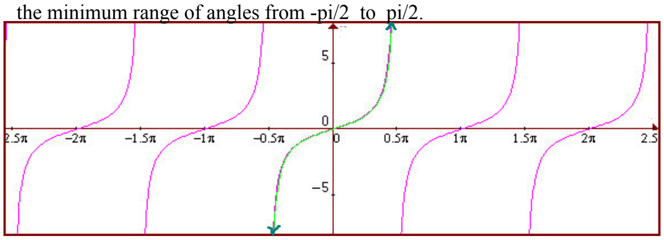
Why is the range of y = arccos x   from 0 to pi?

Since the values of cosine must include all values from -1 to 1, we need a range of angle values that gives us ALL of these cosine values. If we look at the graph of the cosine function, we see that by choosing the angle to range from 0 to pi, we get all these values.



Can you predict the domain and range of the inverse tangent function by looking at the graph of the tangent function? What are the possible tangent values? From -1 to 1? What angle range do we need in order to obtain all possible tangent values?
SEE NEXT PAGE . . .

As we can see from the graph below, the possible tangent values extend from positive to negative infinity. Between each pair of vertical asymptotes, there are identical curves. By choosing only 1 of these curves, we obtain all possible tangent values and the minimum range of angles from -pi/2 to pi/2.



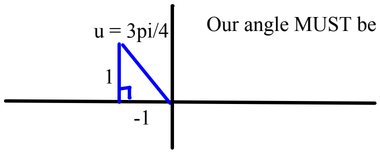
We define inverse tangent in the following way:

y = arctan x   where -inf < x < inf and -pi/2 < y < pi/2 and

WE MAY ALWAYS rewrite y = arctan x   as   tan y = x.

Example: If tan u = -1, and "u" is in quadrant II, find "u" without a calculator AND by using the arctan function on your calculator.   SEE NEXT PAGE ...

If tan u = -1, we may construct a right triangle with our angle in quadrant II .

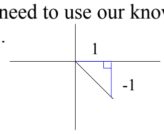


Our angle MUST be u = 3pi/4.

Now, if we use the inverse trig function to solve tan u = -1, we first rewrite this as u = arctan (-1). We then input -1 and use the key on our calculator that looks like $\tan^{-1}$.

The result is u = -0.78539816 = -pi/4. Why didn't we get 3pi/4??????

Answer: If you go back to the specified range of angle values for the inverse tangent function, we see that the angle must be from -pi/2 to pi/2. It is true that tan (-pi/4) IS equal to -1. However, "u" is outside this specified angle range because it is in the second quadrant. We need to use our knowledge of trigonometry to add pi radians to -pi/4 to get u=3pi/4.



Tan (-pi/4) = -1/1 = -1
Tan (3pi/4) = -1 also

Example: If tan u = 0.2, find ALL values of u that satisfy this equation in the angle range where "u" ranges from 0 up to 2pi.

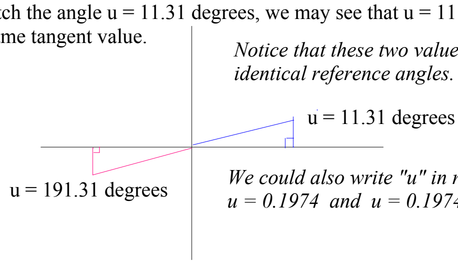
First, we use our calculator to find one solution. Rewrite tan u = 0.2 as

u = arctan 0.2

Enter 0.2 and the arctan key on your calculator to get u = 0.1974 radians or u = 11.31 degrees (answers are rounded).

If we sketch the angle u = 11.31 degrees, we may see that u = 11.31+180 = 191.31 has the same tangent value.

Notice that these two values of "u" have identical reference angles.



We could also write "u" in radian form as u = 0.1974 and u = 0.1974 + pi = 3.3390

Graphs of Inverse Trigonometric Functions

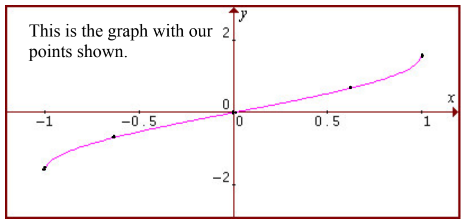
Probable the easiest way to sketch graphs that contain inverse trig functions is to follow these guidelines:

1. Rewrite the inverse trig function as its corresponding non-inverse form.
2. Pick angle values (y-values) in the range of the inverse trig function.
3. Calculate the x-values.
4. Plot the points and draw a smooth curve through.

Example: Graph y = arcsin x

First, rewrite this as x = sin y.
Pick values of y from -pi/2 to pi/2 and calculate x-values.

x	y
-1	-pi/2
-.707	-pi/4
0	0
0.707	pi/4
1	pi/2

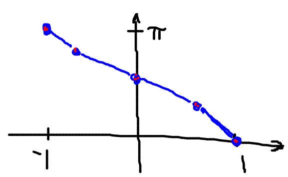


Example: Sketch the graph of y = arccos x

We start by rewriting this as cos y = x where our angle y ranges from 0 up to pi.

We pick y-values from 0 to pi and calculate the x-values. Then we plot these points.

x	y
1	0
0.707	pi/4
0	pi/2
-.707	3pi/4
- 1	pi

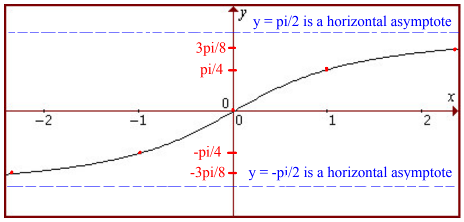


Example: Sketch the graph of y = arctan x

We rewrite this as tan y = x. Here our angle varies from -pi/2 to pi/2. We pick y-values including y= -pi/4, y= 0, y= pi/4, y = 3pi/8, y= -3pi/8, calculate the x-values, and plot these points. We can then generate the arctan curve.
SEE NEXT PAGE . . .

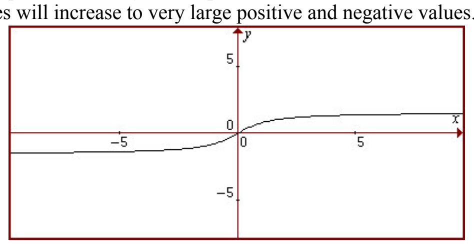
y = arctan x
x = tan y

x	y
-1	-pi/4
0	0
1	pi/4
-2.41	-3pi/8
2.41	3pi/8



Notice that this graph never hits the horizontal lines y = pi/2 and y = -pi/2. These lines are horizontal asymptotes. As the y-values picked are closer and closer to pi/2 or -pi/2, the x-values will increase to very large positive and negative values.

A zoomed-out graph is shown here:



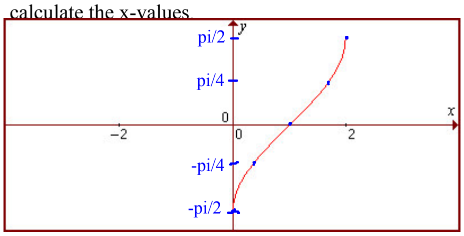
Example: Sketch the graph of f(x) = arcsin(x - 1).

Write this as y = arcsin(x - 1). Rewrite this as x - 1 = sin y.

We may write this as x = 1 + sin y.

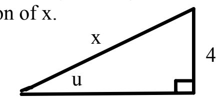
Pick y-values from -pi/2 to pi/2 and calculate the x-values

x	y
0	-pi/2
0.293	-pi/4
1	0
1.707	pi/4
2	pi/2



More Examples Using the Inverse Trig Functions

Example: Given the right triangle below, sse an inverse function to write "u" as a function of x.



Solution: We know that sin u = 4/x.
By the inverse relationship, we may rewrite this as arcsin(4/x) = u.

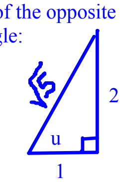
Example: Find the exact value of sin(arctan 2) by using the inverse relationship.

Solution: We let the angle "u" equal the arctan expression to give us

u = arctan 2 OR u = arctan(2/1).

Now sketch a triangle with acute angle "u" such that the tangent of angle "u" is 2/1. SEE NEXT PAGE . . .

Since the ratio of the opposite side over the adjacent side is 2/1, we get the following triangle:



By the Pythagorean Theorem, the hypotenuse of this triangle is the square root of 5.

Since sin u = (opposite)/(hypotenuse),

sin u = $\frac{2}{\sqrt{5}}$

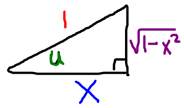
But wait! Since "u" = arctan 2, then sin(arctan 2) = $\frac{2}{\sqrt{5}}$

Example: Write sin(arccos x) as an algebraic expression that does not contain any trig functions in it.

Solution: Again, let "u" = the inverse trig function to result in u = arccos x

Now, rewrite this as x = cos u OR better yet, x/1 = cos u.

Note that the adjacent side is "x" and the hypotenuse is 1. Now, construct a triangle corresponding to x/1 = cos u. SEE NEXT PAGE . . .



$\cos u = \frac{\text{ADJACENT}}{\text{HYPOTENUSE}} = \frac{x}{1}$

$\sin u = \frac{\text{OPPOSITE}}{\text{HYPOTENUSE}} = \frac{\sqrt{1-x^2}}{1}$

By the Pythagorean Theorem, the opposite side is found to be $\sqrt{1-x^2}$.

Since $u = \arccos x$

$\sin(\arccos x) = \sin u = \frac{\sqrt{1-x^2}}{1}$.

We may simplify the above expression to get sin(arccos x) = $\sqrt{1-x^2}$.

Do the Exercises for this Section!!!