Welcome to Section 7.3 - Solving Trigonometric Equations

Solving trigonometric equations is a skill that you will need to use in Calculus and many other science and math courses.

Finding General Solutions to Equations of the form cos x = b, sin x = b, tan x = b

Example: Find all solutions to the equation cos x = 0.

We know that x = pi/2 is a solution and also x = 3pi/2 is a solution.
We must also include ALL coterminal angle solutions.

$$x = \frac{\pi}{2} \pm 2\pi, \frac{\pi}{2} \pm 4\pi, \frac{\pi}{2} \pm 6\pi, \text{etc...}$$

$$x = \frac{3\pi}{2} \pm 2\pi, \frac{3\pi}{2} \pm 4\pi, \frac{3\pi}{2} \pm 6\pi, \text{etc...}$$

Remember that an "integer" is any positive or negative counting number such as 1,-1,2,-2,3,-3, etc. An integer may also = 0.

We may write this in the following way, where "n" is any integer.

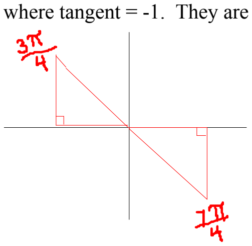
$$x = \frac{\pi}{2} + 2n\pi \quad \text{OR} \quad x = \frac{3\pi}{2} + 2n\pi$$

We may even write this in a more compact form given here, since we have solutions occuring in intervals of pi/2 units.

$$x = \frac{\pi}{2} + n\pi$$

Example: Find all solutions to the equation tan x = -1.

You should (hopefully) remember that the special case 45-45-90 reference triangle has tangent values of 1 or -1. There are two such reference triangles where tangent = -1. They are shown below.



The angle x with tan x = -1 is one with 45 degree reference angle such that sin x and cos x are opposite values. This occurs at 135 degrees and 315 degrees.

The two angles x such that tan x = -1 are x = 3pi/4 and x = 7pi/4.

We may write the general solution for ALL coterminal solutions as

$$x = \frac{3\pi}{4} + 2n\pi$$

$$x = \frac{7\pi}{4} + 2n\pi$$

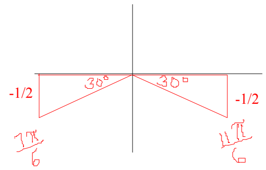
where "n" is an integer.

Since all of these solutions are "pi" units apart, we may write this in even more condensed form as $x = \frac{3}{4}\pi + n\pi$.

Remember that "n" is any positive or negative integer or n = 0.

Example: Find all solutions to the equation $\sin x = -\frac{1}{2}$.

We use our knowledge of the "special case" 30-60-90 triangle that has a sine value of -1/2. A sketch is helpful.



Note that we obtain 7pi/6 by adding pi/6 to pi and we obtain 11pi/6 by subtracting pi/6 from 2pi. Remember that 180 deg = pi, 360 deg = 2pi, and 30 deg = pi/6.

There are two 30-60-90 triangles that have terminal angle such that sin x = -1/2. The two solutions obtained are x = 7pi/6 & x = 11pi/6.

We list ALL solutions by adding multiples of 2pi to each of these. The complete set of solutions is:

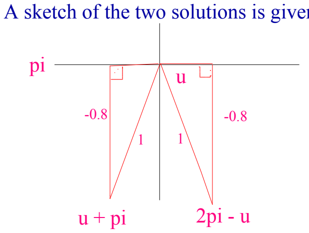
$$x = \frac{7\pi}{6} + 2n\pi$$

$$x = \frac{11\pi}{6} + 2n\pi$$

What if our equation was sin x = -0.8 ? We will tackle this problem next.

Example: Solve the equation sin x = -0.8 .

We have to "think" here. Sine values are negative in the 3rd and 4th quadrants. A sketch of the two solutions is given below.



We can see from the sketch here that if we find the angle in the 4th quadrant, then the other angle in 3rd quadrant is found by adding this reference angle u to pi. Another solution is found by subtracting the reference angle from 2pi.

We find our first solution with the inverse sine function and our calculator.

Two solutions for x will be 2pi - u, & u + pi.

If sin x = -0.8, x = arcsin(-0.8) = -0.9273 (rounded).

Thus, our reference angle would be u = 0.9273 and two solutions are

$x = 2\pi - 0.9273, \; x = \pi + 0.9273$

In decimal form, this would be x = 5.356 & x =4.069 . The general solution is

$x \simeq 5.356 + 2n\pi, \; x = 4.069 + 2n\pi$ where "n" is any integer.

Solving Equations of the Form cos ax = b, sin ax = b, & tan ax = b

Example: Solve sin 4x = 1/2

To solve an equation like sin 4x = 1/2, we use the same method we used to solve for sin x = 1/2, except we solve for 4x instead. We then divide our solutions by 4 to get solutions for x.

Here, we use the fact that sin (pi/6) = 1/2 and sin (5pi/6) = 1/2.

Thus we know that 4x = pi/6 and 4x = 5pi/6 are two solutions.

Accounting for the coterminal angles, we get

$4x = \frac{\pi}{6} + 2n\pi, \quad 4x = \frac{5\pi}{6} + 2n\pi$ where n is an integer.

To solve for x, multiply both sides of the above solutions by 1/4 to get

$$\frac{1}{4}(4x) = \frac{1}{4}\left(\frac{\pi}{6} + 2n\pi\right), \quad \frac{1}{4}(4x) = \frac{1}{4}\left(\frac{5\pi}{6} + 2n\pi\right)$$

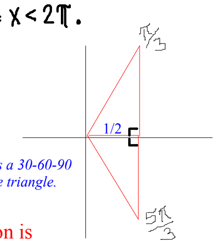
After we multiply through and simplify, we get

$$x = \frac{\pi}{24} + \frac{n\pi}{2}, \quad x = \frac{5\pi}{24} + \frac{n\pi}{2}.$$

Example: Solve cos 2x = 1/2 and list all solutions where $0 \le x < 2\pi$.

We first find the angle 2x such that cos(angle) = 1/2.

The angles that satisfy this are 2x = pi/3, and 2x = 5pi/3.



Note: this is a 30-60-90 special case triangle.

The general solutions are

$2x = \frac{\pi}{3} + 2n\pi$

$2x = \frac{5\pi}{3} + 2n\pi$

Multiply both sides by 1/2 to get

$\frac{1}{2}(2x) = \frac{1}{2}\left(\frac{\pi}{3} + 2n\pi\right)$

$\frac{1}{2}(2x) = \frac{1}{2}\left(\frac{5\pi}{3} + 2n\pi\right)$

$\rightarrow$ The general solution is

$x = \frac{\pi}{6} + n\pi$

$x = \frac{5\pi}{6} + n\pi$

Now, to find all solutions from 0 up to 2pi, let "n" = 0,1,2,3 etc .

If n = 0, the solutions are pi/6 and 5pi/6. If n = 1, the solutions are 7pi/6 & 11pi/6.

If n = 2, the solutions are 13pi/6 & 17pi/6, which is > 2pi, so we stop with n=1.

The complete set of solutions is $\left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \right\}$.

Solving Other Forms of Trigonometric Equations

From the previous examples, we see that it is "fairly" easy to solve a trig equation if it is in the form sin ax = b, cos ax = b, or tan ax = b.

What if it's not in this form?

Answer: Write the equation in this form!

Procedure to Solve an Equation Containing Trigonometric Functions

1. Write the equation in the form sin ax = b, cos ax = b, or tan ax = b, by
   A) Combining terms, performing algebraic operations, and applying trigonometric identities.

2. Use previous methods to solve the equation.

Example: Solve the equation (sin x)(tan x) - sin x = 0.

Here's a big hint: Factor and let each factor = 0 .

See Next Page . . .

Example: Solve the equation (sin x)(tan x) - sin x = 0.

Factor this as sin x(tan x - 1) = 0. Let each factor = 0 to get

sin x = 0 and tan x - 1 = 0. This results in

sin x = 0 and tan x = 1.

These have solutions x = 0, x = pi & x = pi/4, x = 5pi/4.

The general forms are

$$\left.\begin{array}{l} x = 0 + 2n\pi \\ x = \pi + 2n\pi \end{array}\right\} x = n\pi$$

$$\left.\begin{array}{l} x = \frac{\pi}{4} + 2n\pi \\ x = \frac{5\pi}{4} + 2n\pi \end{array}\right\} x = \frac{\pi}{4} + n\pi$$

Note that we are able to "condense" down each pair of solutions that are "pi" units apart. Our final solutions are $x = n\pi,\ x = \frac{\pi}{4} + n\pi$.

Example: Solve the equation $2\sin^2 x - 1 = 0$.

To solve this, we want to obtain the form sin x = b, so we are solving for sin x.

$2\sin^2 x = 1$

$\sin^2 x = \frac{1}{2}$ Now, take square roots of both sides. Remember the +/- sign.

$\sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

$\sin x = \frac{\sqrt{2}}{2}$ $\quad \sin x = \frac{-\sqrt{2}}{2}$ We write the +/- solution in the form of two separate equations.

The angles that solve these equations are from special case 45-45-90 triangles.
If sin x = sqrt(2)/2, x = pi/4 & x = 3pi/4.
If sin x = -sqrt(2)/2, x = 5pi/4 & x = 7pi/4.

The general form is $x = \frac{\pi}{4} + 2n\pi$, $x = \frac{3\pi}{4} + 2n\pi$

$x = \frac{7\pi}{4} + 2n\pi$, $x = \frac{5\pi}{4} + 2n\pi$

Since all of these solutions occur pi/2 units apart, we may write the solution in condensed form as $x = \frac{\pi}{4} + \frac{n\pi}{2}$.

Example: Solve $\cos^2 2x - \cos 2x = 1$ and find all solutions from 0 up to 2pi.

Here, we need to get all the terms to one side first. We obtain

$\cos^2 2x - \cos 2x - 1 = 0$

This is quadratic form! If we let u = cos 2x and substitute, we get the equation

$u^2 - u - 1 = 0$ We solve this equation with the quadratic formula to get

$$u = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$$

The decimal form of these solutions are u = 1.618 & u = -.618 (rounded).

We then let these solutions be equal to cos 2x since u = cos 2x. We obtain

$1.618 = \cos 2x$

$-.618 = \cos 2x$

We're not done yet!
You now have to use your calculator to find values of the angle "2x" for each of these cases. You then have to find the other additional solutions that are associated. Then multiply by 1/2 to get x. SEE NEXT PAGE . . .

Solve the two cases separately:

$1.618 = \cos 2x$

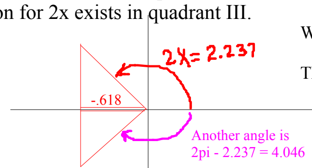
2x = arccos(1.618) = ???????????

There is NO angle with cosine value of 1.618 since all cosine values range from -1 to 1.

$-.618 = \cos 2x$

2x = arccos(-.618) = 2.237 radians

Note that this is between pi/2 = 1.57 radians and pi = 3.1415 radians. Note that another solution for 2x exists in quadrant III.



We get two solutions. 2x = 2.237 and 2x = 4.046

The general solutions are

$$2x = 2.237 + 2n\pi$$
$$2x = 4.046 + 2n\pi$$

We multiply by 1/2 to solve for x.

$$x = \tfrac{1}{2}(2.237) + \tfrac{1}{2}(2n\pi)$$
$$x = \tfrac{1}{2}(4.046) + \tfrac{1}{2}(2n\pi)$$

The general solutions for x are

$$x = 1.119 + n\pi$$
$$x = 2.023 + n\pi$$

What are all solutions from 0 to 2pi? SEE NEXT PAGE

The general solutions for x are

$$x = 1.119 + n\pi$$
$$x = 2.023 + n\pi$$

To find all solutions from 0 to 2pi, let n = 0, 1, 2, and so on until solutions are greater than 2pi.

If n = 0, x = 1.119 & x = 2.023
If n = 1, x = 4.261 & x = 5.165
If n = 2, both values of x are greater than 2pi.

The solutions to the equation $\cos^2 2x - \cos 2x = 1$ **are**

x = 1.119
x = 2.023
x = 4.261
x = 5.165

You may check these all with your calculator by inputting them into the original equation. You will find that these answers will be "off" a little bit because of the rounding that was done.

Example: Solve $1 = \frac{\cos^2 u}{1 - \sin u}$.

Here again, we wish to obtain the form sin u = a or cos u = a. This may be achieved by simplifying the right side by applying an identity.

We may rewrite this as $1 = \frac{1 - \sin^2 u}{1 - \sin u}$ by applying a Pythagorean identity.

Now, factor the numerator as a difference of squares to get $1 = \frac{(1 + \sin u)(1 - \sin u)}{1 + \sin u}$.

Cancel out the factors of (1 + sin u) to get 1 = 1 - sin u .

Now, subtract 1 from both sides to get 0 = -sin u so sin u = 0.

Solutions are u = 0, and u = pi.
Since the general solution is $u = 0 + 2n\pi$, $u = \pi + 2n\pi$ and solutions occur every pi units, we may write the "condensed" form of the solution as $u = n\pi$.

There is a very quick graphing method to solve trigonometric equations. Read on!

Solving Trigonometric Equations by Graphing

To solve a trigonometric equation by graphing, you simply need to
1. Move ALL the terms to one side of = so that the equation is set equal to 0.

2. Let "y" equal the non-zero side of the equation.

3. Graph the equation on a graphing utility and zoom in on the x-intercepts.
   These x-values are the solutions to the original equation since y = 0 at these points.

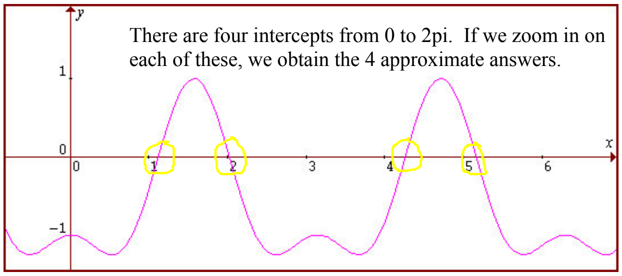
Example: Solve $\cos^2 2x - \cos 2x = 1$ by graphing.

First, move all terms to one side. $\cos^2 2x - \cos 2x - 1 = 0$

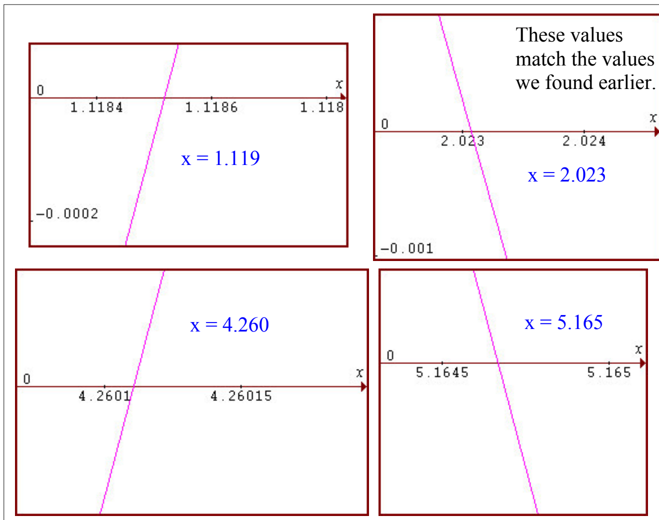
Now, let $y = \cos^2 2x - \cos 2x - 1$ and graph this function.

Then zoom in on each x-intercept. The graph is shown on the next page.

The graph, as generated with Graphmatica, is shown below.



The zoomed in graphs are shown on the next page.



That's all for this section - Do the homework problems!