Welcome to Section 7.5 - Multiple-Angle and Product-to-Sum Formulas

In this section, we use double-angle formulas, half-angle formulas, and power reducing formulas to simplify expressions and solve equations.
We also use product-to-sum and sum-to-product formulas to simplify expressions and solve equations.

Double-Angle Formulas

If we let u=x and v=x in the angle-sum identities, we obtain the following:
sin(x + x) = sin x cos x + cos x sin x
cos(x + x) = cos x cos x - sin x sin x
tan(x + x) = (tan x + tan x)/[1 + (tan x)^2] OR we may write these as

$$\sin 2x = 2\sin x\cos x$$

$$\left.\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 1 - 2\sin^2 x \end{aligned}\right\}$$

Note that we may write the double-angle identity for cos 2x in several forms by noting that (cos x)^2 = 1 - (sin x)^2 and (sin x)^2 = 1 - (cos x)^2.

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Example: Solve the equation given below by using a double-angle identity.

$$\frac{\sin 2x}{\cos x} = \frac{1}{2}$$

Find all solutions in the interval from 0 up to 2pi.

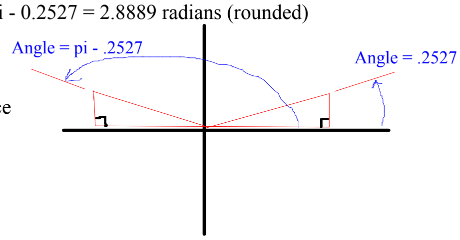
Replace sin(2x) with its equivalent. Then cancel the cos x terms.

$$\frac{2\sin x\cancel{\cos x}}{\cancel{\cos x}} = \frac{1}{2} \rightarrow 2\sin x = \frac{1}{2}$$

Divide both sides by 2 to get $\sin x = \frac{1}{4}$.

One solution is x = arcsin(1/4) = 0.2527 radians (rounded)
Another solution is x = pi - 0.2527 = 2.8889 radians (rounded)

To see how the second solution is obtained, it is helpful to sketch reference triangles.



Example: Verify the identity sin 2x + cos 2x + 1 = 2cos x (sin x + cos x).

Start by replacing cos 2x and sin 2x with their equivalents.

$$\sin 2x + \cos 2x + 1$$
$$= 2\sin x\cos x + 2\cos^2 x - 1 + 1$$
$$= 2\sin x\cos x + 2\cos^2 x$$

Now, just factor out 2cos x.

$$= 2\cos x(\sin x + \cos x).$$

Power-Reducing Identities

If we solve the double-angle identities given below for $\cos^2 x$ & $\sin^2 x$,
we obtain identities called "Power-Reducing" identities. See if you can derive these.

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x$$

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Power-Reducing Identities - Here are the results of solving for the squared functions

$$\sin^2 x = \frac{1 - 2\cos 2x}{2}$$

$$\cos^2 x = \frac{1 + 2\cos 2x}{2}$$

$$\tan^2 x = \frac{1 - 2\cos 2x}{1 + 2\cos 2x}$$

Notice that we may derive the tangent-squared identity by dividing sin^2 by cos^2.

The power-reducing identities may be rewritten by making the following substitutions:

Let 2x = u

Let x = u/2 *Note that since x is 1/2 of 2x, it follows that u/2 MUST = x.*

See if you can make these substitutions and then solve for sin(u/2), cos(u/2), and tan(u/2). The results are given on the next page.

$$\sin^2\left(\frac{u}{2}\right) = \frac{1-\cos u}{2}$$

$$\cos^2\left(\frac{u}{2}\right) = \frac{1+\cos u}{2}$$

$$\tan^2\left(\frac{u}{2}\right) = \frac{1-\cos u}{1+\cos u}$$

Now, take square roots of both sides of these equations to get what are known as the half-angle identities.

Half-Angle Identites

$$\boxed{\begin{aligned} \sin\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1-\cos u}{2}} \\ \cos\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1+\cos u}{2}} \\ \tan\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1-\cos u}{1+\cos u}} \end{aligned}}$$

Note that the value of the +/- sign depends on the quadrant you are in. In the examples that follow, you will see how the value of this sign is determined.

Example: Use the half-angle identities to find the exact value of sine, cosine, and tangent of 112.5 degrees.

Here, we let u/2 = 112.5 degrees. This means that u = 225 degrees, and we may find sin u, cos u, tan u exactly, because 225 is a special-case angle where we know that sin 225 and cos 225 = -[sqrt(2)]/2.

To determine whether to use the + or - sign, note that 112.5 degrees is in Quadrant II, we know that sine is positive, cosine is negative, and tangent is negative.

$$\sin 112.5^\circ = +\sqrt{\frac{1-\cos 225^\circ}{2}} = \sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$$

$$\cos 112.5^\circ = -\sqrt{\frac{1+\cos 225^\circ}{2}} = -\sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

We may multiply write these expressions in a much simpler form by multiplying the numerator and denominator by 2 inside each radical. Also, we can then solve for tan 112.5 by dividing sin 112.5 by cos 112.5 . SEE NEXT PAGE . . .

$$\sin 112.5^\circ = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2} \cdot \frac{(2)}{(2)}} = \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$\cos 112.5^\circ = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2} \cdot \frac{(2)}{(2)}} = -\sqrt{\frac{2-\sqrt{2}}{4}}$$

Divide sin 112.5 by cos 112.5 to get tan 112.5 .

$$\tan 112.5^\circ = -\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

Example: Use the given half-angle identity to verify the identity given below.

$$\tan \frac{u}{2} = \frac{1-\cos u}{\sin u}$$

Start with the identity at the right and simplify the radical. $\tan\left(\frac{u}{2}\right) = \pm\sqrt{\frac{1-\cos u}{1+\cos u}}$

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$$\pm\sqrt{\frac{1-\cos u}{1+\cos u}}$$

$$= \pm\frac{\sqrt{1-\cos u}}{\sqrt{1+\cos u}} \cdot \frac{\sqrt{1+\cos u}}{\sqrt{1+\cos u}}$$

Multiply numerator and denominator by the radical conjugate.

$$= \pm\frac{\sqrt{1-\cos^2 u}}{1+\cos u}$$

Multiply the radicands together in the numerator. The radical disappears in the denominator.

$$= \frac{\pm\sqrt{\sin^2 u}}{1+\cos u}$$

Replace 1 - (cos u)^2 with its equivalent.

$$= \frac{\pm\sin u}{1+\cos u}$$

The radical and square operation cancel each other.

Now, how can we just "drop" the +/- sign????? SEE NEXT PAGE . . .

If u/2 is in Quadrant I, angle u will be in quadrant I or quadrant II and sin u will always be positive. This will give us a positive value for tan (u/2) every time since the denominator will always be positive.

If u/2 is in Quadrant II, angle u will be in Quadrant III or IV and sin u will automatically be negative, resulting in tan(u/2) always being negative, AS IT SHOULD BE.

If u/2 is in Quadrant III, angle u will be in Quadrant I or II. (Think about this, if u/2 = 200 deg, u = 400 deg = 40 deg, if u/2 = 265 deg, u = 530 deg = 170 deg) Sin u will automatically be positive, resulting in tan(u/2) always being positive, AS IT SHOULD BE.

If u/2 is in Quadrant IV, angle u will be in Quadrant III or IV and sin u will automatically be negative, resulting in tan(u/2) always being negative (using the same reasoning of the previous case), AS IT SHOULD BE.

So, we can drop that +/- sign and show that $\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$.

Example: Verify that cos u cos v = (1/2)[cos(u - v) + cos(u + v)].

Work with the more complex right side by replacing cos(u - v) and cos(u + v) with their equivalents by using the angle sum and difference formulas. The desired result quickly follows.

$$\frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$
$$= \frac{1}{2}\left[\cos u \cos v + \sin u \sin v + \cos u \cos v - \sin u \sin v\right]$$
$$= \frac{1}{2}\left[\cos u \cos v + \cos u \cos v\right]$$
$$= \frac{1}{2}\left[2\cos u \cos v\right]$$
$$= \cos u \cos v$$

Similarly, we verify identities for sin u sin v, sin u cos v, and cos u sin v. These are called "Product-to-Sum" identities and are given on the next page.

Product-to-Sum Identities

sin u sin v = (1/2)[cos(u - v) - cos(u + v)]

sin u cos v = (1/2)[sin(u - v) + sin(u + v)]

cos u cos v = (1/2)[cos(u - v) + cos(u + v)]

cos u sin v = (1/2)[sin(u + v) - sin(u - v)]

Example: Rewrite sin(3x)cos x using one of the above identities.

Here, we would let 3x represent u and let x represent v to get

sin(3x)cos x
= (1/2)[sin(3x - x) + sin(3x + x)]
= (1/2)[sin(2x) + sin(4x)]
OR 0.5sin(2x) + 0.5sin(4x)

In Calculus, the integral of sin(3x)cos x is impossible to evaluate as it is, whereas the integral of the quantity 0.5sin(2x) + 0.5sin(4x) is easy to evaluate.

MORE IDENTITIES!

There is one last bunch of identities that are useful at times. These are the Sum-To-Product Identities. One of 4 is given below.

$$\cos x + \cos y = 2\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

This identity may be proved by applying the product-to-sum identity on the right side. You must let (x - y)/2 represent u and let (x + y)/2 represent v.

$$2\left[\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right]$$

$$= 2\left[\frac{1}{2}\left[\cos\left[\frac{x-y}{2} - \frac{x+y}{2}\right] + \cos\left[\frac{x-y}{2} + \frac{x+y}{2}\right]\right]\right]$$

$$= \cos\left(\frac{-2y}{2}\right) + \cos\left(\frac{2x}{2}\right)$$ *2 cancels with 1/2. We combine the fractions and simplify.*

$$= \cos(-y) + \cos x$$ *Apply the identity cos u = cos(-u).*

$$= \cos y + \cos x$$ *This is known as a "sum-to-product" identity. A complete list is given on the next page.*

Sum-to-Product Identities

$$\cos x + \cos y = 2\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

$$\sin x + \sin y = 2\cos\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

All of these identities may be verified with a method similar to that used in verifying the first identity.

Example: Solve the equation sin 2x + sin 4x = 0

We apply the sum-to-product identity to get
sin 2x + sin 4x = 2 sin[(2x + 4x)/2 cos (2x - 4x)/2]
= 2sin(3x)cos(-x)
= 2sin(3x)cos x *since cos(-x) = cos x*

We let this equal 0 to get
0 = 2 sin(3x)cos x.

Divide both sides by 2 to get
0 = sin(3x)cos x.

Let each factor equal zero to get 0 = sin(3x) and 0 = cos x.

The solutions are 3x = 0 + 2npi, 3x = pi + 2npi which results in x = 0 + (2npi)/3 , x = pi/3 + (2npi)/3. This may be written as

x = (npi)/3

ALSO
x = pi/2 + 2npi, x = 3pi/2 + 2npi. This may be written as

x = pi/2 + npi

Example: Verify the identity $\frac{\cos u + \cos v}{\sin u + \sin v} = \cot\left(\frac{u+v}{2}\right)$.

Apply the sum-to-product identity to the left side to get

$$\frac{\cos u + \cos v}{\sin u + \sin v} = \frac{2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)}{2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)}$$

$$= \frac{\cos\left(\frac{u+v}{2}\right)}{\sin\left(\frac{u+v}{2}\right)}$$

Cancel factors of 2 and cancel the identical cosine factors.

$$= \cot\left(\frac{u+v}{2}\right)$$

Apply the identity (cos x)/(sin x) = cot x

That's all for this section. Do the homework!