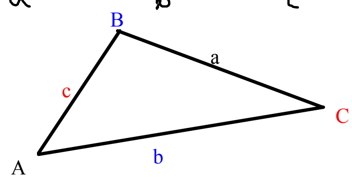
Section 8.1 - Law of Sines

It may be shown that for ANY triangle with sides of length a, b, & c and corresponding angles A, B, and C that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Note: We can also write this as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example: A triangle has angles A=30 deg, B=40 deg, and side a=10" . Solve this triangle.

By subtraction, angle C has measure 110 degrees, since all 3 angles add to 180. We can solve this triangle using the Law of Sines. SEE NEXT PAGE.

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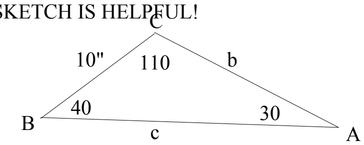
A SKETCH IS HELPFUL!



By the law of sines

10/sin(30) = c/sin(110)
20 = c/(0.939692)
20(0.939692) = c = 18.79" rounded

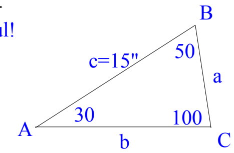
10/sin(30) = b/sin(40)
20 = b/(0.642788)
20(0.642788) = b = 12.86" rounded

A "good" sketch is helpful since it allows you to "estimate" what the answers should be.

The answers obtained match what you would expect from the sketch.

Example: Given that a triangle has angles A = 30 deg & B = 50 deg, and side length c = 15", solve the triangle.

A "good" sketch is helpful!
By subtraction, angle C is 100 degrees.

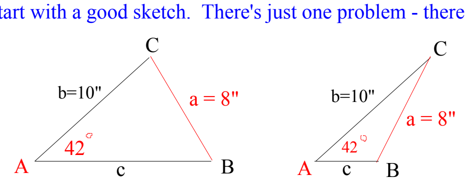


By the law of sines, b/sin 50 = 15/sin 100, so
b = (sin 50)(15)/sin 100 = 11.67 " rounded.

a/sin 30 = 15/sin 100, so
a = (sin 30)(15)/sin 100 = 7.62 " rounded.

Example: A triangle has angle A = 42 deg, side a = 8", and side b = 10".
Solve this triangle.

Start with a good sketch. There's just one problem - there are two possibilities!



What we have to do here is solve for one of the possibilities using the law of sines.
We then must use our knowledge of trigonometry to find the second solution.

By the law of sines, (sin B)/10 = (sin 42)/8.

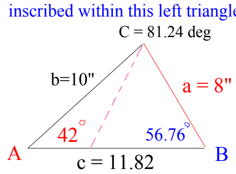
This results in Sin B = 10(sin 42)/8 = 0.836413257 . . .
Using the arcsin function, we get B = arcsin(0.8364132. . .) = 56.76 deg (rounded).

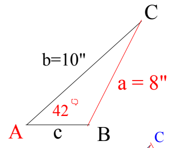
By subtraction, C = 180 - (42 + 56.76) = 81.24 deg. CONTINUED . . .

Now find side c. By the law of sines, c/sin(81.24) = 8/(sin 42) or

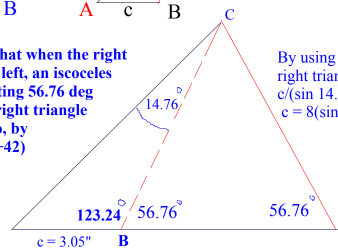
c = 8(sin 81.24)/(sin 42) = 11.82 (rounded)

We notice that this side length matches our triangle on the left. Also notice that the right triangle is inscribed within this left triangle with the addition of the dashed line.





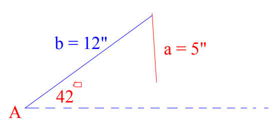
The figure to the right shows that when the right triangle is inscribed inside the left, an iscoceles triangle is formed. By subtracting 56.76 deg from 180 deg, we see that the right triangle has angle B = 123.24 deg. Also, by subtraction, C = 180 - (123.24+42) or C =14.76 deg.



By using the law of sines on the right triangle, we get c/(sin 14.76) = 8/(sin 42). c = 8(sin 14.76)/(sin 42) =3.05"

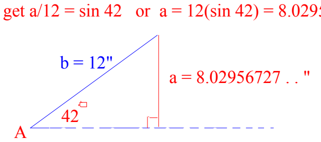
Example: A triangle has angle A = 40 deg, side a = 5", and side b = 12". Solve this triangle.

Start with a good sketch. In this case, it is clear that there is NO triangle that can be formed since side a=5" is too short.



THERE IS NO SOLUTION IN THIS CASE.

Note that the shortest that side "a" may be is the side length "a" that forms a right triangle. In this case. we get a/12 = sin 42 or a = 12(sin 42) = 8.02956727 . . . (See figure below)



Try to find side "a" by using Law of Sines! - You will get an error message!

Also note: If side "a" is between 8.02956 . ." and 12", two triangles may be formed.

If side "a" is MORE THAN 12", NO TRIANGLE may be formed.

Using Law of Sines in the Three Cases

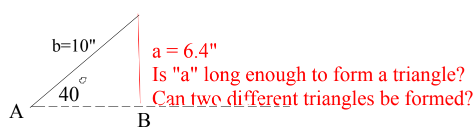
CASE 1 Side-Angle-Side (ASA) - When two angles and the side between the two angles are given, there is always a single solution that may be obtained by using the Law of Sines. Note that the 3rd angle is obtained by subtracting the sum of the given angles from 180 degrees. A sketch is helpful.

CASE 2 Angle-Angle-Side (AAS) - When two angles and the side opposite one of the angles are given, there is always a single solution that may be obtained by using the Law of Sines. A sketch is helpful.

CASE 3 Side-Side-Angle (SSA) - When two sides and the angle NOT formed by the two sides are given, there may be 1, 2, or NO solutions. You should ALWAYS make a careful sketch showing the angle given along with the two given side lengths. If it appears from the sketch that there are two (or no) possibilities for placing the second side length, calculate the length of a second side length that forms a right triangle and compare the given length to this length that forms a right triangle. Also, if there is NO solution, you will get an error message if you try to use Law of Sines.
SEE EXAMPLE ON NEXT PAGE

Example: A triangle has side "a" = 6.4", angle A = 40 deg, and side "b" = 10 ". Solve the triangle if possible.

From the sketch, we see that it is not clear whether or not there is a solution.



If side "a" were to form a right triangle, it would have to satisfy the right triangle relationship
Sin 40 = a/10 or a = 10(sin 40) = 6.427876 . . ."

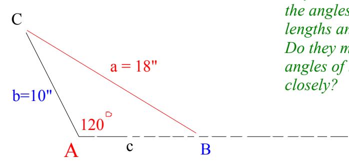
Since side "a" is only 6.4", it is not long enough to form a right triangle, or any triangle.

THERE IS NO SOLUTION.

NOTE: If we try to find angle B with the Law of Sines, we get
(sin B)/10 = (sin 40)/6.4 or sin B = 10(sin 40)/6.4 = 1.00435
NO ANGLE HAS A SINE OF 1.00435 . An error results if we take inverse sine of 1.00435 .

Example: A triangle has side "a" = 18", angle A = 120 deg, and side "b" = 10 ".
Solve the triangle if possible.

Start with a sketch.



As you obtain solutions for all of the angles and sides, plug in these lengths and angles into the sketch. Do they match the lengths and angles of the sketch relatively closely?

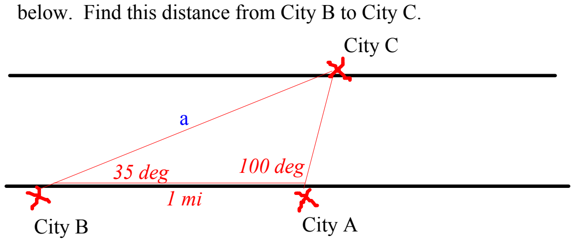
In this case, there is only one possibility for completing the triangle with side "a".
Using Law of Sines results in (sin B)/10 = (sin 120)/18 or

sin B = 10(sin 120)/18 = .4811
Using the inverse sine function results in B = 28.76 degrees.
By subtraction, C = 180 - (120 + 28.76) = 31.24 deg.

If we apply the Law of Sines again, we get c/(sin 31.24) = 18/(sin 120) or
c = (sin 31.24)(18)/(sin 120) = 10.78"

Note that if "a" were 10" or less, NO triangle would be possible! Trying to use the Law of Sines would result in an error.

Example: Cities A,B, & C are on opposite sides of a river. A person wishes to find the distance from City B to City C by taking two measurements as shown below. Find this distance from City B to City C.



The distance from City A to City B is 1 mile.
By subtraction, the angle at City C is 180 - (100+35) = 45 degrees.

Let the distance from City B to City C = "a".
By the Law of Sines, a/(sin 100) = 1/(sin 45) or

a = (sin 100)(1)/(sin 45) = 1.4 miles (rounded)

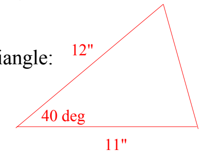
Area of a ANY Triangle

It may be shown that the area of ANY triangle with angles A,B, & C and corresponding sides a, b, and c is given by the formula

Area = (1/2)ab(sin C) = (1/2)bc(sin A) = (1/2)ac(sin B)

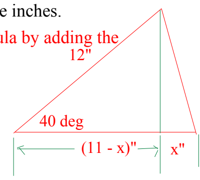
This formula only requires that you have two sides and the angle in between the two sides.

Example: Find the area of this triangle:



By the formula, Area = (1/2)(11)(12)(sin 40) = 42.424 square inches.

Can you find the area of this triangle without using this formula by adding the areas of the two triangles shown here?



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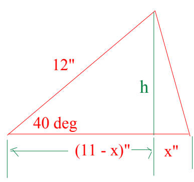
The altitude of this triangle may be found to be equal to h = 12 (sin 40).



Adding the two areas of the two RIGHT triangles results in a total area of
(1/2)(11 - x)(h) + (1/2)(x)(h)
= (1/2)(h)[(11 - x) + x]
= (1/2)(h)[11]

Note that for any triangle, the area formula is (1/2)(base)(height).

Now substitute in h = 12(sin 40) to get
Area = (1/2)(12)(sin 40)(11).

This is the same as the area obtained with the formula.

That's all for this Section - Do the Homework!

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