Section 8.3 Vectors in the Plane

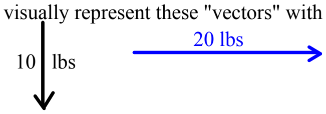
A vector is a mathematical quantity that has both magnitude and direction.

Example: A force of 10 pounds is exerted straight down towards the surface of the earth. Another force of 20 pounds is exerted parallel to surface of the earth from left to right.

Here our first force has a magnitude of 10 pounds with a direction straight down.

Our second force has a magnitude of 20 pounds with a direction toward the right parallel with the surface of the earth.

We can visually represent these "vectors" with "directed line segments".

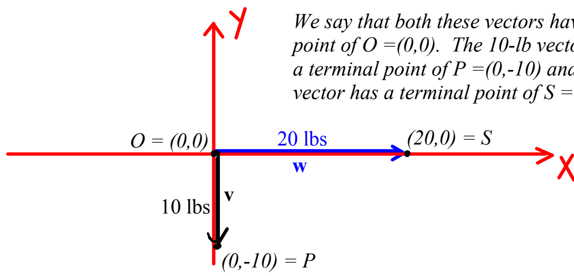


Note that the arrows are drawn with lengths proportional to the magnitudes - the 10 lb arrow is half the length of the 20 lb arrow.

Notation for Vectors

In the previous example, two directed arrows were drawn to represent the vectors given. We may position these two vectors on the xy-axis so that their initial points are at the origin.

We say that both these vectors have initial point of O =(0,0). The 10-lb vector has a terminal point of P =(0,-10) and the 20-lb vector has a terminal point of S = (20,0).

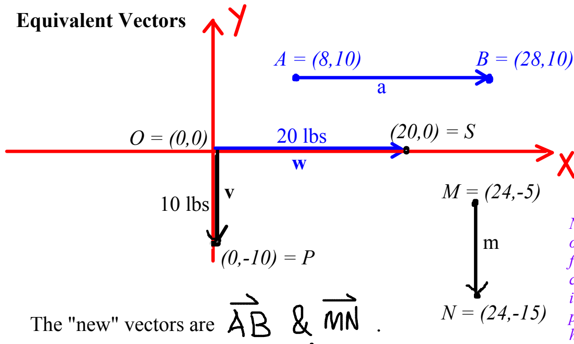


We can call the 10-lb vector $V = \overrightarrow{OP}$.

We can call the 20-lb vector $W = \overrightarrow{OS}$.

Guess what? We may move these vectors to new locations and get equivalent vectors.
SEE NEXT PAGE . . .

Equivalent Vectors



Note that the length of the vectors may be found by comparing coordinates of the initial and terminal points. a & w both have length 20. v &m both have length 10.

The "new" vectors are $\vec{AB}$ & $\vec{MN}$.

We may designate these $a = \vec{AB}$ & $m = \vec{MN}$.

When two vectors have the same magnitude and direction, the vectors are "equivalent". In this example, v is equivalent to m & w is equivalent to a.

From this example, we see that we may always move a vector and obtain an equivalent vector, provided the magnitude (length) and the direction (angle) is unchanged.

Component Form

In the previous example we saw that when the vector was in a position such that initial point was the origin (0,0), the we were able to determine the direction and magnitude from the terminal point alone.

Contrast this to the vectors moved so the initial point was NOT (0,0). For these vectors we needed both initial and terminal points to determine the magnitude and direction.

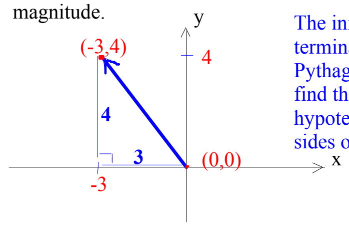
If the initial point of a vector is (0,0), the vector is said to be in "standard position".

Vectors in standard position may be given in terms of their terminal point only (since the initial point is always the origin) as $v = \langle v_1, v_2 \rangle$ where (v_1, v_2) is the terminal point.

This form, $v = \langle v_1, v_2 \rangle$ is known as the "component form".

Example: Sketch the vector with component form < -3,4 > and find its magnitude. SEE NEXT PAGE . . .

Example: Sketch the vector with component form $< -3,4 >$ and find its magnitude.



The initial point is (0,0) and the terminal point is (-3,4). By the Pythagorean Theorem, we may find the magnitude since it is the hypotenuse of a right triangle with sides of length 4 and 3.

Magnitude = $\sqrt{3^2+4^2} = 5$.

Note again: The magnitude is the length of the arrow.

Magnitude of the Component Form

From the example above, we may apply the Pythagorean Theorem to ANY vector in component form $v = \langle v_1, v_2 \rangle$

to say that the magnitude of v is $\sqrt{(v_1)^2+(v_2)^2}$.

Notation for Magnitude

We indicate the magnitude of a vector v as $\|v\|$.

For example, if $v = < -3, 4 >$, $\|v\| = 5$.

For ANY vector v in component form

$v = \langle v_1, v_2 \rangle$,

$\|v\|$ **= the magnitude** $= \sqrt{(v_1)^2+(v_2)^2}$.

Vectors NOT in Component Form

Example: Sketch the vector v with initial point (1,5) and terminal point (2,-3). Also, find its magnitude.

SEE NEXT PAGE . . .

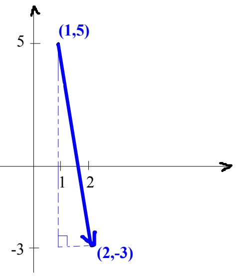
Example: Sketch the vector v with initial point (1,5) and terminal point (2,-3).
Also, find its magnitude.

We can construct a right triangle with base of 1 unit and height of 8 units.

The magnitude may be found by using the Pythagorean Theorem to get a magnitude of

$$||v|| = \sqrt{1^2 + 8^2}$$

$$||v|| = \sqrt{65} \approx 8.06$$



What we are really doing is using the DISTANCE FORMULA for the segment connecting (1,5) to (2,-3).

The DISTANCE FORMULA for the segment connecting (x_1, y_1) to (x_2, y_2)

is $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

For the vector v, the magnitude is equal to the distance = $\sqrt{(2-1)^2 + (-3-5)^2}$.

The Magnitude of ANY Vector with Initial Point P = (p1,p2) and Terminal Point Q= (q1,q2)

If v is a vector with initial point P = (p1,p2) and terminal point Q = (q1,q2), then the magnitude of v is

$$||v|| = \sqrt{(q_2 - p_2)^2 + (q_1 - p_1)^2} .$$

Note: We are simply applying the Pythagorean Theorem

Writing ANY Vector in Component Form

In the last example, we found the magnitude of the vector v with initial point (1,5) and terminal point (2,-3). We may also "shift" this vector so that has an initial point of (0,0), but still has the magnitude and direction of the given vector v.

If you plot the vector v, we see that in order to obtain an equivalent vector with initial point (0,0), we must shift this vector LEFT ONE, and DOWN FIVE.

The coordinates of the NEW terminal point will be reduced by 1 in the x-coordinate and reduced by 5 in the y-coordinate.

Try sketching the vector v along with the new component form vector.

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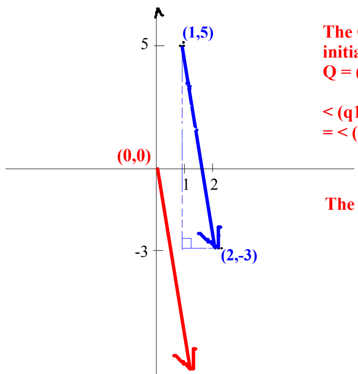


The Component Form of the vector v with initial point P = (p1, p2) and terminal point Q = (q1, q2) will ALWAYS be

< (q1 - p1), (q2-p2) >
= < (terminal x - initial x), (terminal y - initial y) >.

The component form is < 2-1 , -3-5 > = < 1, -8 >.

(1,-8) = New Terminal Point = (2-1, -3-5)

Notice that the direction of the component form has not changed and the magnitude is $\sqrt{1^2 + 8^2} = \sqrt{65}$.

If two vectors have the same component form, we say that they are EQUAL.

Unit Vectors and Zero Vectors

A "Unit Vector" is vector v in component form that has magnitude = 1.

Example: The vector v = < 1, 0 > is a unit vector since the magnitude of v is 1.

Example: The vector v = < 3/5, 4/5 > is a unit vector since its magnitude is

$$\|v\| = \sqrt{\left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2}$$
$$= \sqrt{\tfrac{9}{25} + \tfrac{16}{25}}$$
$$= \sqrt{\tfrac{25}{25}} = 1$$

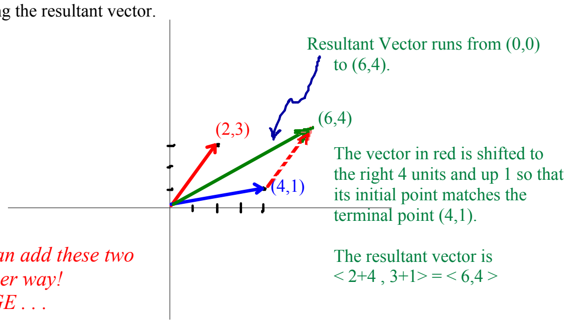
A "Zero Vector" is the vector v = < 0, 0 >. There is only ONE zero vector. The magnitude of the zero vector is 0. Also, ONLY the zero vector has magnitude = 0.

Vector Operations

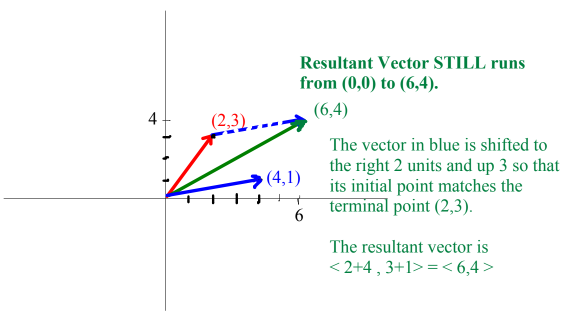
We may add two vectors by shifting one of the vectors (either one) so that it's initial point is equal to the others terminal point. The resultant vector will have the initial point the first vector and the terminal point of the second.

It is easy to see how this works from an example.

Example: Add the vectors v = < 2, 3 > and w = < 4, 1 > by sketching both, moving one and then sketching the resultant vector.



But wait! We can add these two vectors in another way!
SEE NEXT PAGE . . .



Adding Vectors by Adding Components

In the previous example, we saw that when we added < 2,3 > to < 4,1 >, we ended up with a "resultant" vector of < 2+4, 3+1 >.

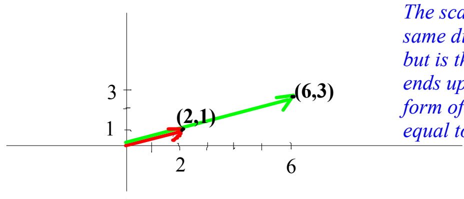
> IN GENERAL,
> if u = < u1, u2 > and v = < v1, v2 >,
> then u + v = < u1+v1, u2+v2 >

Scalar Multiples of Vectors

A "scalar" is a quantity that has ONLY magnitude. In plain English, a "scalar" is simply an ordinary real number.

If a vector is multiplied by a scalar, its magnitude is multiplied by the scalar.

Example: Sketch $u = \langle 2, 1\rangle$ and also sketch $3u$. (Here our scalar is "3")



The scalar multiple has the same direction as $\langle 2, 1 \rangle$ but is three times longer and ends up having a component form of $\langle 6, 3 \rangle$ which is equal to $\langle 3x2 , 3x1 \rangle$.

Can you think of a rule for multiplying a scalar by a vector?
SEE NEXT PAGE . . .

Multiplying a Vector by a Scalar

When multiplying a vector $v = \langle v1, v2 \rangle$ by a scalar "c", multiply EACH component of v by c.

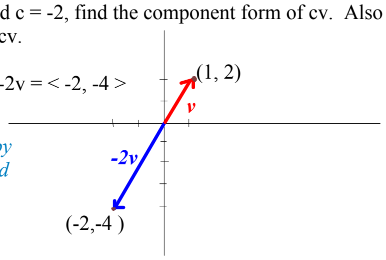
In other words, $cv = c\langle v_1, v_2\rangle = \langle cv_1, cv_2\rangle$.

The Effect of Multiplying a Vector by a Negative Scalar

If we multiply a vector by a negative scalar, we still follow the above rule. The effect on the vector, however, is to reverse the direction 180 degrees.

Example: If $v = \langle 1, 2 \rangle$ and $c = -2$, find the component form of cv. Also sketch both v and cv.

$cv = -2 \langle 1, 2 \rangle = -2v = \langle -2, -4 \rangle$



The effect of multiplying v by -2 reverses the direction and doubles the magnitude.

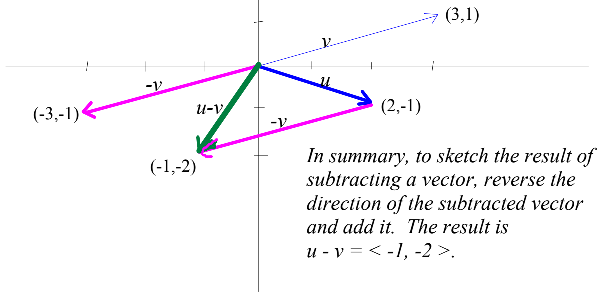
Subtracting a Vector from a Vector

In order to fully understand vector subtraction, you must first accept the following vector property:

u - v = u + (-1)v

Example: Sketch the result of subracting u - v where u = < 2, -1 > and v = < 3 , 1>.

Sketch u, sketch v, and then sketch (-1)v. Note (-1)v = -v = < -3, -1>.
Add u to -v by moving the initial point of -v to the terminal point of u.



In summary, to sketch the result of subtracting a vector, reverse the direction of the subtracted vector and add it. The result is u - v = < -1, -2 >.

Rule for Subtracting Component Forms

The last example demonstrated the following rule:

If u = < u1, u2 >, and v = < v1, v2 > ,
then u - v = < u1-v1 , u2-v2 >.

General Rule for Adding or Subtracting Scalar Multiples of Vectors

We may combine the rules for scalar multiplication and vector addition and subtraction to get the following rules:

If u = < u1, u2 >, v = < v1, v2 >, and c & d are scalars, then

$$cu + dv = c\langle u_1, u_2\rangle + d\langle v_1, v_2\rangle = \langle cu_1 + dv_1, cu_2 + dv_2\rangle .$$

Also

$$cu - dv = c\langle u_1, u_2\rangle - d\langle v_1, v_2\rangle = \langle cu_1 - dv_1, cu_2 - dv_2\rangle .$$

Examples of these rules are on the following page.

Example: If u = < 2, -4 > and v = < 1, 5 >,
find 2u+3v and also find 4u-5v.

2u + 3v = 2 < 2, -4 > + 3 < 1, 5 >
= < 4, -8 > + < 3, 15 >
= < 4 + 3 , -8 + 15 >
= < 7 , 7 >

4u - 5v = 4 < 2, -4 > - 5 < 1, 5 >
= < 8, -16 > - < 5, 25 >
= < 8 - 5 , -16 - 25 >
= < 3 , -41 >

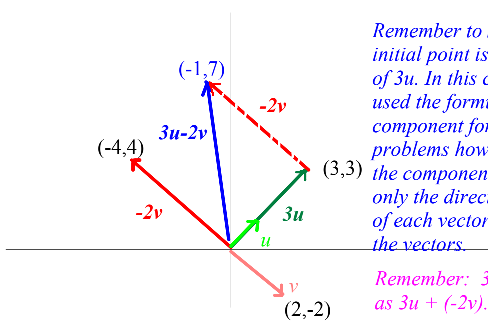
Example: If u = <1 , 1> and v = < 2, -2 >, sketch the vector w = 3u - 2v.

First note that 3u - 2v = 3u + (-2)v where -2v = -2 <2, -2> = < -4, 4 >.

Also, 3u = 3 < 1, 1 > = < 3, 3 >.

So, sketch 3u = < 3, 3> and -2v = < -4, 4 >. Then add them.

SEE NEXT PAGE.



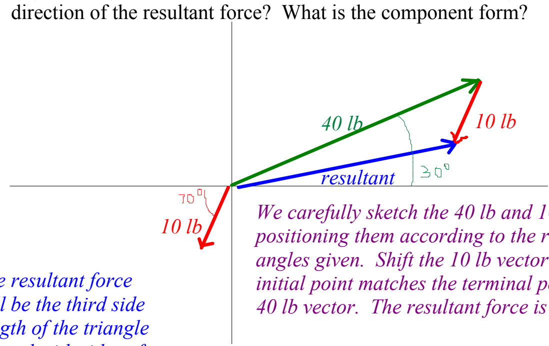
Remember to shift -2v so that its initial point is at the terminal point of 3u. In this case we could have also used the formula to obtain the final component form < -1, 7 >. In many problems however, we are not given the component forms to work with but only the direction and magnitude of each vector so we must sketch the vectors.

Remember: 3u - 2v is the same as 3u + (-2v).

Example: If a force of 40 pounds is exerted to the right 30 degrees above the positive x-axis and a second force of 10 pounds is exerted leftward at an angle 20 degrees below the negative x-axis, what is the magnitude and direction of the resultant force? What is the component form?

Start with a sketch!!!
SEE NEXT PAGE . . .

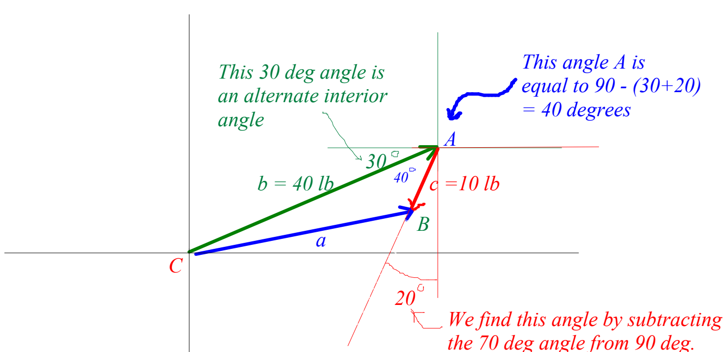
Example: If a force of 40 pounds is exerted to the right 30 degrees above the positive x-axis and a second force of 10 pounds is exerted leftward at an angle 70 degrees below the negative x-axis, what is the magnitude and direction of the resultant force? What is the component form?



We carefully sketch the 40 lb and 10 lb vectors positioning them according to the respective angles given. Shift the 10 lb vector so that its initial point matches the terminal point of the 40 lb vector. The resultant force is shown.

The "magnitudes" of these vectors are the pounds of force for each. The magnitude of the resultant, its direction, and its component form may be found with the Law of Cosines and some additional trigonometry.

The resultant force will be the third side length of the triangle formed with sides of length 10 and 40.
SEE NEXT PAGE . . .



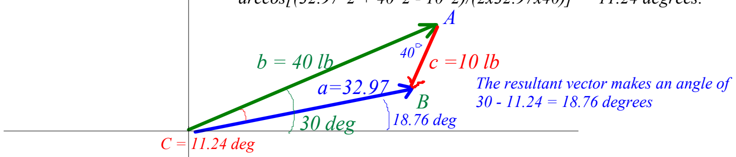
We can turn this into a Law of Cosines problem by relabling the triangle as ABC with sides b = 40, c = 10, and angle A = 40 deg. As you can see in the figure above, there are a few steps involved in finding the angle A.

By the Law of Cosines, a = $\sqrt{40^2+10^2-2(40)(10)(\cos 40^\circ)} = 32.97$

Now, find the angle that the resultant vector makes with the x-axis and find its component form.
See NEXT PAGE . . .

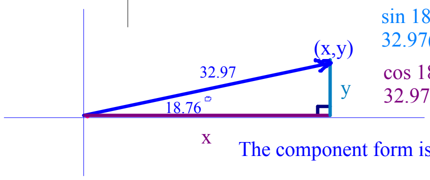
Since the 40 lb vector makes a 30 deg angle with the x-axis, the angle made by the 32.97 lb vector will be equal to 30 - (angle C).

By the Law of Cosines, C = arccos[(32.97^2 + 40^2 - 10^2)/(2x32.97x40)] = 11.24 degrees.



The resultant vector makes an angle of 30 - 11.24 = 18.76 degrees

To find the component form of the resultant vector, we apply right triangle trigonometry (see below).



sin 18.76 = y/32.97
32.97(sin 18.76) = y = 10.60

cos 18.76 = x/32.97
32.97(cos 18.76) = x = 31.21

The component form is < x,y > = < 31.21,10.60 >

Standard Unit Vectors

A "unit vector" is a vector with magnitude = 1.

Example: < 0.6, 0.8 > is a unit vector since the magnitude is

$\sqrt{(0.6)^2 + (0.8)^2} = 1$

Example: < 1, 0 > is a unit vector since its magnitude is $\sqrt{1^2 + 0^2} = 1$.

The "Standard" Unit Vectors are < 1, 0 > and < 0, 1 > .

We designate these as i = < 1, 0 > and j = < 0, 1>.

Note: We may write ANY vector as a combination of multiples of the standard unit vectors i and j.

Example: Write the vector < 2, -4 > as a combination of i and j.

< 2, -4 > = < 2, 0 > + < 0, -4 >
= 2 < 1, 0 > + (-4) < 0, 1>
= 2i + (-4)j OR 2i - 4j

Example: Write the component form of -3i + 4j.

-3i + 4j
= -3 < 1, 0 > + 4 < 0, 1 >
= < -3, 0 > + < 0 , 4 >
= < -3, 4 >

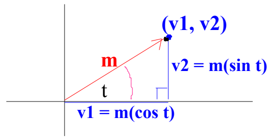
Trigonometric Form of a Vector

If v = < v1, v2 >, and vector u makes an angle "t" with the positive x-axis and has magnitude , "m" we may represent u as

v = < m(cos t), m(sin t) > .

If we factor out the magnitude "m", we get
v = m < cos t, sin t >.

We can also write this in "i-j" form as
v = m[(cos t)i + (sin t) j]



Note: cos t = (v1)(m)
m(cos t) = v1

sin t = (v2)(m)
m(sin t) = v2

Trigonometric Form of a Unit Vector

If u is a unit vector (magnitude = 1), its trigonometric form is simply

u = (cos t) i + (sin t) j

Notes on Notation

In many texts, the symbol θ is used instead of t. The trigonometric form of u would be given as

u = (cos θ) i + (sin θ) j

Also, if v is any vector, the magnitude is often given as ||v|| ***and***

v = ||v|| [(cos θ) i + (sin θ) j]

Direction Angle

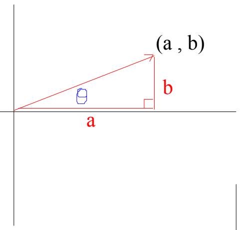
Given that a vector "v" makes angle θ with the positive x-axis, and v = ai + bj, then tan(θ) = b/a. WHY IS THIS TRUE? SEE NEXT PAGE. .

Direction Angle

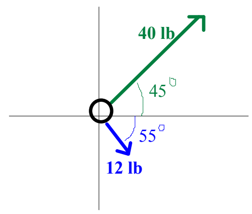
Given that a vector "v" makes angle θ with the positive x-axis, and $v = ai + bj$, then $\tan(\theta) = b/a$.

Since $v = ai + bj$,
$v = \langle a, 0 \rangle + \langle 0, b \rangle$
$= \langle a, b \rangle$

According to right-triangle trigonometry, $\tan \theta = b/a$

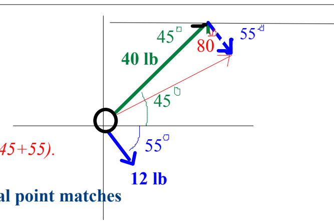


Example: Two forces act on a ball as shown here. What is the direction and magnitude of the resultant force?



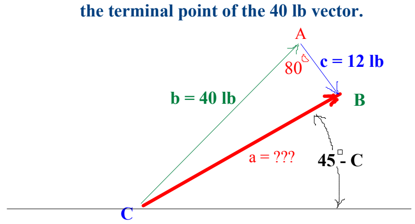
Looks like a Law of Cosines problem!
SEE NEXT PAGE . . .

Example: Two forces act on a ball as shown here. What is the direction and magnitude of the resultant force?



The 80 degree angle is found by subtracting 180 - (45+55).

We first translate the 12 lb vector so that its initial point matches the terminal point of the 40 lb vector.



The magnitude of the vector may be found by using the Law of Cosines.

We may labeling the triangle ABC and let the magnitude = side "a".

By the Law of Cosines

$a^2 = 40^2 + 12^2 - 2(40)(12)(\cos 80)$
$a = 39.72$ lbs

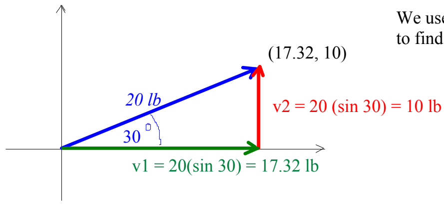
The angle that the resultant vector makes if found by subtracting (45 - angle C).

By the Law of Cosines, angle C = $\arccos[(39.72^2 + 40^2 - 12^2)/(2 \times 39.72 \times 40)] = 17.31$ deg

The resultant vector has angle 45 - 17.31 = 27.69 degrees

Example: Given that a 20 lb force is exerted at an angle of 30 degrees with the positive x-axis, find the horizontal and vertical component vectors that result in this force when added together. Write in component and trigonometric form.

We may represent this force as a vector with magnitude = 20 and angle of 30 deg. We also may sketch in the vertical and horizontal component vectors.



We use right triangle trig to find the x and y components.

This vector may be represented as v = < v1, v2 >, where v1 = 20 sin 30 and v2 = 20 cos 30.

The component form is < 17.32, 10 >.

The trigonometric form is v = 20[(cos 30)i + (sin 30)j]
= 20[0.866 i + 0.5 j]
= 17.32 i + 10j

That's all for this section. Do the homework problems!

Note: The i-j form of v = < v1, v2 > may always be written v = (v1)i + (v2)j.