Section 8.4 - Vectors and Dot Products

Dot Product of Two Vectors

The dot "product" of two vectors u and v where u = < u1, u2 > and v = < v1, v2 >, is defined as

$$u \cdot v = u_1 v_1 + u_2 v_2$$

Note: The dot product is a scalar, not a vector. It has magnitude but no direction.

Example: If u = < 2, 4 > and v = < -1, 0 >, find the dot product of u and v.

$$u \cdot v = 2 \cdot (-1) + 4 \cdot 0$$
$$= -2 + 0$$
$$= -2$$

Example: If u = < 1, 2 > and v = < -4, 2 >, find the dot product of u and v.

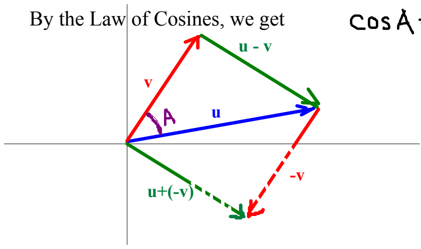
$u \cdot v =$ 1(-4) + 2(2) = -4 + 4 = 0

Dot Product Used to Find the Angle Between 2 Vectors

If vectors u = < u1, u2 > and v = < v1, v2 > are given, we see from the figure below that a triangle is formed with vectors u,v, and (u-v). The side lengths of this triangle are given by the magnitudes of these vectors. The angle between the vectors u and v is given as angle A.

By the Law of Cosines, we get

$$\cos A = \frac{\|u\|^2 + \|v\|^2 - \|u-v\|^2}{2\|u\|\|v\|}$$



Substitute in the quantities

$$\|u\| = \sqrt{u_1^2 + u_2^2}$$
$$\|v\| = \sqrt{v_1^2 + v_2^2}$$
$$\|u-v\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}$$

The result, after algebraically simplifying is $\cos A = \frac{u \cdot v}{\|u\|\|v\|}$.

Note: Numerator is a DOT PRODUCT of u & v.

We may also write this as $\|u\|\|v\| \cos A = u \cdot v$.

Example: If u = < 3, 1 > and v = < 4, -2 >, find the angle formed by these vectors.

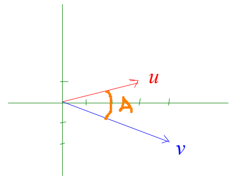
Using the formula $\cos A = \dfrac{u \cdot v}{\|u\|\|v\|}$

results in

$$\cos A = \frac{(3)(4)+(1)(-2)}{\sqrt{3^2+1^2}\sqrt{4^2+(-2)^2}}$$

$$= \frac{10}{\sqrt{10}\sqrt{20}}$$

$$= \frac{10}{\sqrt{200}} \approx .707$$



Using the inverse cosine function results in A = 45 degrees.

Note that the angle "A" returned will always be less than or equal to180 degrees.

Orthoganal Vectors

"Orthoganal" vectors are any two vectors u & v such that the dot product of u and v is zero. $u \cdot v = 0$

Since $\|u\|\|v\| \cos A = u \cdot v$

letting the dot product = 0 means that $\|u\|\|v\| \cos A = 0$.

This means that if two vectors u and v are orthoganal, then
u = < 0, 0 > or
v = < 0, 0 > or
cos A = 0.

If both u and v are both non-zero and u and v are orthoganal, then cos A = 0.
This means the angle A is 90 degrees and u and v are perpendicular to each other.

Example: Are the vectors u = < 3, 2 > and v = < -2, 4 > orthoganal?

The dot product is (3)(-2) + (2)(4) = -6 + 8 = 2. Since the dot product is NOT zero, these vectors are NOT orthoganal. Also, the angle A between them will NOT be 90 degrees.

Example: Are the vectors u = < 0, 0 > and v = < 1,9> orthoganal?

Here the dot product is 0(1) + 0(9) = 0. u and v ARE orthoganal.

Example: Are the vectors u = < 2, 3 > and v = < -6,4 > orthoganal?

The dot product is 2(-6) + 3(4) = 0. u and v ARE orthoganal. Furthermore, since u and v are both nonzero, cos A = 0 and the angle A between them is A = 90 degrees.

Parallel Vectors

Two vectors are parallel if the angle "A" between them is A = 0 or A = 180 deg. This occurs if one vector "v" is a scalar multiple of the other vector "u". In other words, u = < u1, u2 > and
v = cu = c < u1, u2 > = < cu1, cu2 >.

$$\cos A = \frac{u \cdot v}{\|u\| \|v\|}$$

If you plug in the components into you obtain cos(A) = 1, or cos(A) = -1 so A = 90 or 180 deg.

Just remember

Only vectors that are scalar multiples of each other are parallel

Example: Are the vectors u = 2i + 4j and v = -20i - 40j parallel?

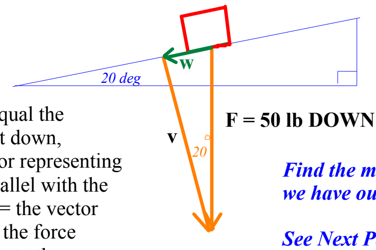
The component forms are u = < 2, 4 > and v = < -20, -40 >.
Since v = -10 < 2, 4 > = -10u, v is a scalar multiple of u and the vectors are parallel.

Note: If you found the angle A between these two vectors, you would obtain cos A = -1 and A = 180.

Applications of Vectors - Finding Components of Force

In Physics applications, it is often necessary to find the vertical and horizontal components of a force exerted at angle A.

Example: A 50# box is on a 20 degree ramp. What is the force parallel to the ramp that is needed to prevent the box from sliding down the ramp?



Note that the force vectors form a right triangle that is similar to the one formed by the ramp.

We let "F" equal the force straight down, w = the vector representing the force parallel with the ramp, and v = the vector representing the force perpendicular to the ramp.

Find the magnitude of "w" and we have our answer!

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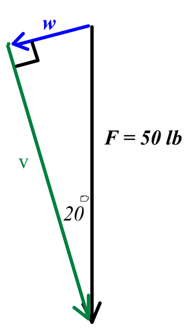
The force needed to prevent the box from sliding will have the same magnitude as "w" but will be in the opposite direction.



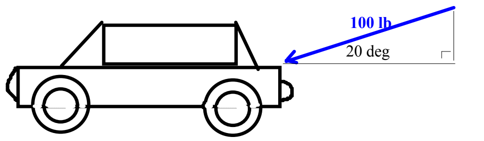
Using right-triangle trigonometry, we have that sin(20) = w/50.

This results in w = 50(sin 20) = 17.1 pounds.

Thus, a force of 17.1 lbs in the opposite direction would be required to prevent the box from sliding. In reality, that force would probably be friction, however if the ramp were slippery, some other force (perhaps a person) would be needed to push at the box up the ramp.

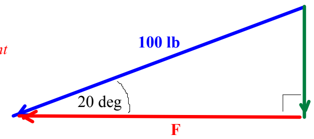
Note that we could also determine the magnitude of the vector "v" using right triangle trigonometry.

Example: A force of 100 pounds exerted at a 20 deg angle (as shown below) is required to move a car that is on a level driveway. If the car is moved 10 feet, what is the work performed?



We must find the horizontal component of the 100 lb force since this component of the force is in the direction of the motion. It is labeled "F" below.

Notice that the horizontal and vertical component vectors "add" to form the 100 lb vector.



By applying right triangle trigonometry, we get $\cos(20) = F/100$ and $F = 100(\cos 20) = 93.97$ lb.

The work performed is (93.97)(100) = 939.7 foot-pounds

That's all for this section - DO THE HOMEWORK!!!!