Section 8.5 - Trigonometric Form of a Complex Number

In this section, we see how it is possible to write a complex number in the same trigonometric form used for vectors. We then will see how it is possible to obtain nth roots of complex numbers by using this form.

Complex Numbers & Their Representation

A complex number "z" has both real AND imaginary components and may always be written in the standard form

z = a + bi where "a" is the real part and "bi" is the imaginary part.

Remember that $i = \sqrt{-1}$ and $i^2 = -1$.

Example: 4 + 5i is a complex number where a = 4 and b = 5i.

Example: Write the product of (2 - 3i)(4 + i) in standard form.

(2 - 3i)(4 + i) = 8 + 2i - 12i - 3i^2
= 8 - 10i - 3(-1)
= 11 - 10i OR 11 + (-10i)

Example: Is the number 2 a complex number, even though there is no i-part?

The answer is "yes". We may write 2 as 2 + 0i. "2" is a complex number with an imaginary part = 0. When the imaginary part = 0, the complex number is usually referred to as a "real" number.

Example: Is the number 3i a complex number, even though there is no real-part?

The answer again is "yes". We may write 3i as 0 + 3i. "3i" is a complex number with a real part = 0. When the real part = 0, the complex number is sometimes referred to as a "pure imaginary" number.

The Complex Plane

If we let the the x-axis represent the value of "a" from a + bi and we let the y-axis represent the value of "b" from a+bi, we obtain what is called "The Complex Plane".

Example: Plot the point that represents -3 + 2i on the complex plane.

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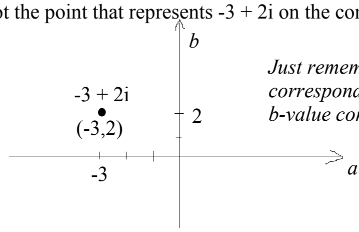


Just remember that the a-value corresponds to the x-axis and the b-value corresponds to the y-axis.

Absolute Value of a Complex Number

The "absolute value" of a complex number is defined as the distance that the complex number is from the origin (0,0) when plotted on the complex plane.

We apply the distance formula to the distance from (a,b) to (0,0) to get

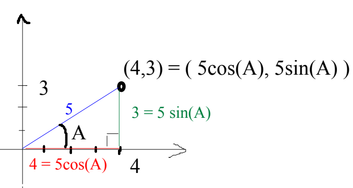
$$|a+bi| = \sqrt{a^2+b^2} .$$

Note that if b=0, the absolute value is simply the positive part of the real number "a", which is what you have learned before.

Trigonometric Form of a Complex Number

If we plot the complex number 4 + 3i in the complex plane, we could find that this complex number has absolute value $\sqrt{4^2+3^2} = \sqrt{25} = 5$.

We see that the x-component may be written as 4 = 5cos(A) and the y-component may be written as 3 = 5 sin(A) where "A" is the angle made with the positive x-axis.



Thus, the complex number 4 + 3i may be written as 5cos(A) + 5sin(A) i since a = 5cos(A) and b = 5 sin(A). We could also write this as 4 + 3i = 5[cos A + i(sin A)] where the absolute value of 4 + 3i = 5. In general, we may write ANY complex number in the form

$$\boxed{\begin{aligned} a+bi &= |a+bi|(\cos A + i \sin A) \\ &= \sqrt{a^2+b^2}(\cos A + i \sin A) \end{aligned}}$$

Trigonometric Form Definitions

We have the following from the previous page:

$$\boxed{\begin{aligned} a+bi &= |a+bi|(\cos A + i\sin A) \\ &= \sqrt{a^2+b^2}\,(\cos A + i \sin A) \end{aligned}}$$

There is a special name for the absolute value of a+bi. We call this absolute value the "modulus" and typically designate it as "r".

We call the angle "A" the "argument" where again, "A" is the angle made with the positive x-axis by the segment connecting (a,b) to (0,0). Thus, we can write

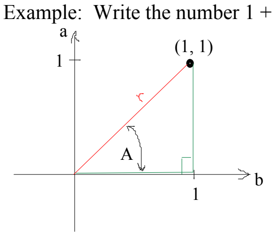
$$\boxed{\begin{aligned} &a+bi = r(\cos A + i\sin A) \\ &\text{WHERE } r = \sqrt{a^2+b^2} \end{aligned}}$$

Writing a Given Complex Number in Trigonometric Form

To write a complex number in trigonometric form, you need to do the following:

1. *Plot the number a+bi = (a,b) in the complex plane.*
2. *Determine the angle "A" made by the segment from (a,b) to (0,0). This is the "argument". Note that tan A = b/a.*
3. *Determine the absolute value of a+bi. This value is known as the modulus r.*
4. *Trigonometric form is r[(cos A) + i (sin A)].*

Example: Write the number 1 + i in trigonometric form.

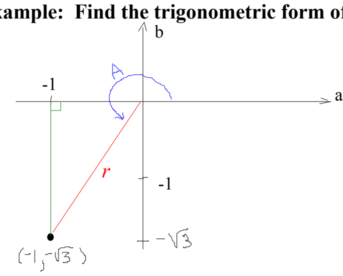


We plot 1 + 1i where a = 1, b = 1.
Tan(1/1) = A, so A= 45 deg OR pi/4.
r = modulus = $\sqrt{2}$.

Trigonometric form is $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$.

Note: We could also write this in degree-form as
$\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$

Example: Find the trigonometric form of $-1-\sqrt{3}$.



We plot the complex number in the complex plane.

Note that tan A = $\frac{-\sqrt{3}}{-1}$

If we use the inverse tan function, we get A = 60 deg BUT THIS IS NOT CORRECT!

Use the fact that "A" is in Quadrant III and the reference angle is 60 deg. This gives us A = 180+60 = 240 deg.

The modulus r is $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

Note: The modulus "r" is the hypotenuse of the right triangle formed.

The trigonometric form is $2(\cos 240^\circ + i \sin 240^\circ)$.

Multiplying and Dividing Complex Numbers

You may be wondering "Why use the trigonometric form?". The answer to this is the fact that the trigonometric forms of complex numbers may be easily multiplied and divided.

If $c_1 = r_1(\cos A_1 + i \sin A_1)$, $c_2 = r_2(\cos A_2 + i \sin A_2)$

Then the product of the two complex numbers is

$$c_1 c_2 = [r_1(\cos A_1 + i \sin A_1)][r_2(\cos A_2 + i \sin A_2)]$$

$$= r_1 r_2 (\cos A_1 + i \sin A_1)(\cos A_2 + i \sin A_2)$$

Now, FOIL this out.

$$= r_1 r_2 [\underset{F}{\cos A_1 \cos A_2} + \underset{O}{\cos A_1 (i \sin A_2)} + \underset{I}{i \sin A_1 \cos A_2} + \underset{L}{i \sin A_1 (i \sin A_2)}]$$

Factor out "i" from the i-terms.

$$= r_1 r_2 [\cos A_1 \cos A_2 + i(\cos A_1 \sin A_2 + \sin A_1 \cos A_2) + i^2 \sin A_1 \sin A_2]$$

Let i-squared = -1. Regroup terms.

$$= r_1 r_2 [\cos A_1 \cos A_2 - \sin A_1 \sin A_2 + i(\cos A_1 \sin A_2 + \sin A_1 \cos A_2)]$$

Now, apply sum and difference formulas for Sine and Cosine. *SEE NEXT PAGE . . .*

$$r_1 r_2 [\cos A_1 \cos A_2 - \sin A_1 \sin A_2 + i(\cos A_1 \sin A_2 + \sin A_1 \cos A_2]$$

$$= r_1 r_2 [\cos(A_1 + A_2) + i \sin(A_1 + A_2)]$$

In other words, to multiply two complex numbers that are in trigonometric form,
1. Multiply the modulus of one by the modulus of the other.
2. Add the arguments.

We get the following result:

$$\boxed{\begin{array}{l} \text{IF } c_1 = r_1(\cos A_1 + \sin A_1) \\ \quad\; c_2 = r_2(\cos A_2 + \sin A_2) \\ \text{THEN } c_1 c_2 = r_1 r_2 [\cos(A_1 + A_2) + i \sin(A_1 + A_2)] \end{array}}$$

We also get a rule for division. The derivation uses trigonometric identities also but will not be shown here. Extra credit perhaps? *SEE NEXT PAGE . . .*

Formula For Divsion of Two Complex Numbers

$$\boxed{\begin{array}{l} \text{IF } c_1 = r_1(\cos A_1 + \sin A_1) \\ \quad\; c_2 = r_2(\cos A_2 + \sin A_2) \\ \text{THEN } \dfrac{c_1}{c_2} = \dfrac{r_1}{r_2} [\cos(A_1 - A_2) + i \sin(A_1 - A_2)] \end{array}}$$

To divide two complex numbers that are in trigonometric form,
1. Divide the modulus of numerator by the modulus of the denominator.
2. Subtract the argument of the denominator from the argument of the numerator.

Example: Multiply c1 = 2(cos 30 + i sin 30) by c2 = 3(cos 60 + i sin 60)

c1 x c2 = (2 x 3)[(cos(30 + 60) + i(sin 30 + 60)]
= 6[cos 90 + i sin 90]
= 6[0 + i(1)] = 6i

Example: Find (1 + i)(1 + i) by converting (1 + i) to trigonometric form.

From a previous example, we found that

$$1 + i = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$$

Thus, $(1 + i)(1 + i) = \sqrt{2}\sqrt{2}\left[\cos(45^\circ + 45^\circ) + i(\sin(45^\circ + 45^\circ)\right]$

$$= 2[\cos 90^\circ + i \sin 90^\circ)$$
$$= 2i$$

Notice from this example that when we square 1 + i, we simply square the modulus and double the argument. We may extend this idea to get a formula for powers of complex numbers.

Powers of Complex Numbers

$$\boxed{\begin{array}{l} \text{IF } C_1 = r_1[\cos A_1 + i \sin A_1] \\ \text{THEN } (C_1)^n = (r_1)^n[\cos(n \cdot A_1) + i \sin(n \cdot A_1)] \end{array}}$$

Example: Find $(1+i)^{10}$.

This could be done algebraically by multiplying out the 10 factors of 1+i but the method using the trigonometric form is MUCH easier.

$$1 + i = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$$

$$(1+i)^{10} = (\sqrt{2})^{10}\left[\cos(10 \times 45) + i \sin(10 \times 45)\right]$$

= 32[cos 450 + i sin 450]
= 32[cos 90 + i sin 90] *Note: 450 deg is coterminal with 90 deg.*
= 32[0 + i(1)]
= 32i

In this example, we see that a tenth root of 32i is u=1+i since u^10 = 32i. We will use the trigonometric form to find the nth root of any complex number. SEE NEXT PAGE . . .

Definition of an "nth" Root of a Complex Number

Let c = r(cos A + i sin A) be a complex number.

"u" is an "nth" root of c if u^n = c.

Example u = i is a "2nd" root of -1 since i^2 = -1.
u = -i is another "2nd" root of -1 since (-i)^2 = -1.

The nth Roots of a Complex Number

The nth roots of c = r(cos A + i sin A) are found by doing the following:

1. Each nth root will have a modulus equal to $\sqrt[n]{r}$,
   so you must take the nth root of r.
2. Divide the argument "A" by n - this gives the argument of one nth root.
3. Divide (A + 360 deg) by n - this gives another argument.
4. Divide (A + 720 deg) by n - this gives another argument.
5. Keep adding multiples of 360 to "A" and then divide by n until you get "n" different arguments, each resulting in another nth root.

Note: A, A+360, A+720, etc are all coterminal, yet they result in different nth roots, all of which result in the complex number "c" when taken to the nth power.

The nth Root Formula

The nth roots of c = r(cos A + i sin A) consist of

$$u_1 = \sqrt[n]{r}\left(\cos\frac{A}{n} + i\sin\frac{A}{n}\right)$$

$$u_2 = \sqrt[n]{r}\left(\cos\frac{(A+360°)}{n} + i\sin\frac{(A+360°)}{n}\right)$$

$$u_3 = \sqrt[n]{r}\left(\cos\frac{(A+2(360°))}{n} + i\sin\frac{(A+2(360°))}{n}\right)$$

•

•

•

$$u_n = \sqrt[n]{r}\left(\cos\frac{(A+(n-1)360°)}{n} + i\sin\frac{(A+(n-1)360°)}{n}\right)$$

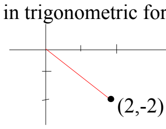
Note: This formula is really the same as what is outlined on the previous page - it's just written as a formula here.

IF THIS DOESN'T MAKE SENSE, DON'T WORRY, SOME EXAMPLES WILL FOLLOW . . .

Example: Find ALL the 5th roots of c = 2 - 2i. Plot these roots.

Our FIRST step is to write this in trigonometric form.
The modulus r is

$r=\sqrt{2^2+2^2}$
$=\sqrt{8}$



The argument is angle "A" where tan(A) = -2/2 = -1 and "A" is in quadrant IV.

"A" is equal to 315 degrees.

Find the 5th root of r. We could write this as $(\sqrt{8})^{\frac{1}{5}} = (8^{\frac{1}{2}})^{\frac{1}{5}} = 8^{\frac{1}{10}}$.
The decimal form would be 1.23114 (rounded).

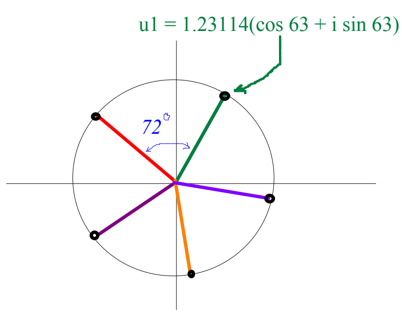
Now, find the arguments (angles) of the different nth roots by dividing each of 315, 315+360, 315+720, 315+1080, 315+1395 by 5.

315/5 = 63 deg
675/5 = 135 deg
1035/5 = 207 deg
1395/5 = 279 deg
1755/5 = 351 deg

ALL 5 ROOTS ARE SHOWN ON THE NEXT PAGE

u1 = 1.23114(cos 63 + i sin 63)
u2 = 1.23114(cos 135 + i sin 135)
u3 = 1.23114(cos 207 + i sin 207)
u4 = 1.23114(cos 279 + i sin 279)
u5 = 1.23114(cos 351 + i sin 351)

All of these complex numbers are plotted here. Notice that they are equally spaced 360/5 = 72 degrees apart. This leads up to a "shortcut" !!



Shortcut for Finding nth Roots

To find the nth roots of a complex number, do the following:
1. Find the 1st nth root by taking the nth root of the modulus and dividing the argument (angle) by n.
2. Add 360/n to get the second nth root argument.
3. Add 360/n to the argument of the second nth root to get the third nth root.
4. Add 360/n to the argument of the third nth root to get the fourth nth root.
5. Keep adding 360/n to get the arguments of all the nth roots.

Of course you will have to write your complex number in trigonometric form first!

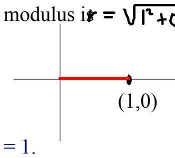
Example: Find all the 6th roots of 1. Plot all of these.

We know that 1 and -1 are nth roots of 1 since (1)^6 = 1 and (-1)^6 = 1.
What about the 4 other 6th roots?

START by writing 1 in trigonometric form. The modulus is $r = \sqrt{1^2+0^2} = 1$.

The argument is "A" = 0 degrees.

The trigonometric form is 1(cos 0 + i sin 0).



The argument of each of the 6th roots = 1^(1/6) = 1.

We take 360/6 = 60 degrees. This means that all 6
of the 6th roots will be 60 degrees apart on the complex plane.

The first 6th root has argument = 0/6 = 0 deg.
The next root has argument = (0 + 360)/6 = 60 deg.
Add 60 to get 120 deg for the next root.
Add 60 to get 180 deg for the next root.
Add 60 to get 240 deg for the next root.
Add 60 to get 300 deg for the last root.

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The 6th complex roots of 1 are

u1 = 1(cos 0 + i sin 0)
= 1(1 + 0) = 1

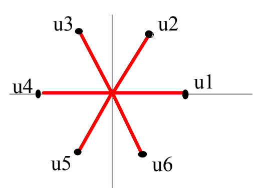
u2 = 1(cos 60 + i sin 60)
$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$

u3 = 1(cos 120 + i sin 120)
$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

u4 = 1(cos 180 + i sin 120)
= 1(-1 + 0) = -1

u5 = 1(cos 240 + i sin 240)
$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

u6 = 1(cos 300 + i sin 300)
$= \frac{1}{2} - \frac{\sqrt{3}}{2} i$



*Just remember: Find the first root.
Then keep adding multiples of 360/n
to find the argument (angle) of all
of the other roots.*

***That's all for this section.
Make sure to do all of the
assigned homework!***