

The Algebra of Functions

Given two functions, $f(x)$ and $g(x)$, we use shorthand notation to indicate the operations of adding, subtracting, multiplying, and dividing the formulas of these two functions. Furthermore, we have shorthand notation to indicate that the entire formula of one function is inputted into the other function. These operations, known as Algebra of Functions, are described below.

Notation For Adding, Subtracting, Multiplying, and Dividing Functions:

$$(f + g)(x) = f(x) + g(x)$$

Example: If $f(x) = 3x + 2$ and $g(x) = x^2$, then $(f + g)(x) = 3x + 2 + x^2$

So, you simply add the two formulas together.

$$(f - g)(x) = f(x) - g(x)$$

Example: If $f(x) = 3x + 2$ and $g(x) = x^2$, then $(f - g)(x) = 3x + 2 - x^2$

So, you simply subtract the formulas.

$$(f g)(x) = f(x) \cdot g(x)$$

Example: If $f(x) = 3x + 2$ and $g(x) = x^2$, then $(fg)(x) = (3x + 2) \cdot x^2$

So, you simply multiply the two formulas together.

$$(f / g)(x) = f(x) / g(x), \text{ provided } g(x) \neq 0$$

Example: If $f(x) = 3x + 2$ and $g(x) = x^2$, then $(f / g)(x) = (3x + 2) / x^2$

So, you simply divide the formulas. This result is not defined if the denominator is zero however.

Example: If $f(x) = 3x - 11$, and $g(x) = x^2 - 1$, find each of the following:

- $(f + g)(x)$
- $(g - f)(2)$
- $(f g)(1)$
- $(f / g)(1)$

$(f + g)(x) = (3x - 11) + (x^2 - 1)$ after applying the definition for addition of functions.
 $= x^2 + 3x - 12$ after combining like terms

$(g - f)(2) = g(2) - f(2)$ after applying the definition for subtraction of functions.
 $= (2^2 - 1) - (3 \cdot 2 - 11)$ after inputting the value $x=2$.
 $= 3 - (-5) = 8$ after simplifying.

$(f g)(1) = f(1) \cdot g(1)$ after applying the definition for multiplication of functions.
 $(3 \cdot 1 - 11) \cdot (1^2 - 1)$ after inputting the value $x=1$.
 $= (-8)(0) = 0$ after simplifying.

$(f / g)(1) = f(1) / g(1)$ after applying the definition for division of functions.
 $(3 \cdot 1 - 11) / (1^2 - 1)$ after inputting the value $x=1$.
 $= (-8) / (0)$ after simplifying.
 $= \text{UNDEFINED}$, since we are dividing by zero.

Composition of Functions

When we input the entire formula from one function in for each x-value of the formula of a second function, we have what is called a **composition of functions**.

Definition of Composition of Functions

$$(f \circ g)(x) = f(g(x))$$

We replace each x in the formula of f(x) with the entire formula of g(x).

Example: If $f(x) = x^2 - x$ and $g(x) = x - 4$, find the following:

- $(f \circ g)(x)$
- $(g \circ f)(x)$
- $(f \circ g)(3)$
- $(f \circ f)(x)$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{by the definition of composition.} \\ &= f(x - 4) && \text{after replacing } g(x) \text{ with its formula.} \\ &= (x - 4)^2 - (x - 4) && \text{after inputting } x-4 \text{ into the function } f(x). \\ &&& \textit{This may be simplified more, if needed.}\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{by the definition of composition.} \\ &= g(x^2 - x) && \text{after replacing } f(x) \text{ with its formula.} \\ &= (x^2 - x) - 4 && \text{after inputting } x^2 - x \text{ into the function } g(x).\end{aligned}$$

$$\begin{aligned}(f \circ g)(3) &= f(g(3)) && \text{by the definition of composition.} \\ &= f(3 - 4) && \text{after replacing } g(3) \text{ with its equivalent.} \\ &= f(-1) && \text{after simplifying } 3-4. \\ &= (-1)^2 - (-1) && \text{after inputting } -1 \text{ into the function } f(x) \\ &= 2 && \text{after simplifying}\end{aligned}$$

$$\begin{aligned}(f \circ f)(x) &= f(f(x)) && \text{by the definition of composition.} \\ &= f(x^2 - x) && \text{after replacing } f(x) \text{ with its formula.} \\ &= (x^2 - x)^2 - (x^2 - x) && \text{after inputting } x^2 - x \text{ into the function } f(x). \\ &&& \textit{This may be simplified more, if needed.}\end{aligned}$$

A Special Composition: Inverse Functions

A very special composition is one where two functions f(x) and g(x) are defined so that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. **When this is the case, we say that f(x) and g(x) are inverse functions of each other.**

Example: $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x + 1}$

These functions are inverses of each other since both $(f \circ g)(x)$ and $(g \circ f)(x)$ equal x.