Quadratic Equations in One Variable

Definition
A quadratic equation in x is any equation that may be written in the form
$ax^2 + bx + c = 0$, where a, b, and c are coefficients and $a \neq 0$.

Note that if a=0, then the equation would simply be a linear equation, not quadratic.

Examples
$x^2 + 2x = 4$ is a quadratic since it may be rewritten in the form $ax^2 + bx + c = 0$ by applying the Addition Property of Equality and subtracting 4 from both sides of =.

$(2 + x)(3 – x) = 0$ is a quadratic since it may be rewritten in the form $ax^2 + bx + c = 0$ by applying the Distributive Property to multiply out all terms and then combining like terms.

$x^2 - 3 = 0$ is a quadratic since it has the form $ax^2 + bx + c = 0$ with b=0 in this case.

$3x^2 – 2/x + 4 = 0$ is not a quadratic since it has the term 2/x. The term 2/x is the same as $2x^{-1}$, and quadratics do not have x raised to any power other than 1 or 2.

Just remember: *Quadratics always have an x^2 term*, possibly an x-term, and possibly a constant term! If your equation has an *x^2 term* or will have an *x^2 term* after multiplying out, it may be a quadratic, provided the other terms fit the form.

Solving Quadratic Equations – Method 1 - Factoring

The easiest way to solve a quadratic equation is to solve by factoring, *if possible*.
Here are the steps to solve a quadratic by factoring:

1. **Write your equation in the form $ax^2 + bx + c = 0$** by applying the Distributive Property, Combine Like Terms, and apply the Addition Property of Equality to move terms to one side of =.
2. **Factor your equation** by using the Distributive Property and the appropriate factoring technique. Note: Any type of factoring relies on the Distributive Property.
3. **Let each factor = 0 and solve**. This is possible because of the Zero Product Law.

Example: Solve $(3x + 4)x = 7$
$(3x + 4)x = 7$ Given
$3x^2 + 4x = 7$ by the Distributive Property
$3x^2 + 4x – 7 = 0$ by the Addition Property of Equality

Now, factor $3x^2 + 4x – 7 = 0$
This factors as $(3x + ?)(x - ?) = 0$ or $(3x - ?)(x + ?) = 0$ where the two unknown numbers multiply to -7 when we use the Distributive Property to multiply out. Also the first two terms must multiply out to $3x^2$. The middle products must add up to 4x.

$(3x + 7)(x - 1) = 0$ gives us middle products 7x and –3x adding up to 4x.

By the Zero Product Law, we can state
$3x + 7 = 0$ and $x-1 = 0$.

Solve these two equations by using the Addition Property of Equality and the Division Property of Equality.
$3x + 7 = 0 \rightarrow 3x = -7 \rightarrow$ $x = -7/3$
$x - 1 = 0 \rightarrow$ $x = 1$

Solving Quadratic Equations – Method 2 – Extracting Square Roots

Extracting square roots is a very easy way to solve quadratics, provided the equation is in the correct form.

Basically, Extracting Square Roots allows you to rewrite $x^2 = k$ as $x = \pm\sqrt{k}$, where k is some real number. Algebraically, we are taking square roots of both sides of the equation as shown below and inserting the $\pm$ to account for both a positive and negative case. Note that the squared quantity must be isolated on one side of = before you can extract the square roots.

Example: Solve $x^2 = 9$ by extracting square roots

$x^2 = 9$	Given
$\sqrt{x^2} = \pm\sqrt{9}$	Extract Square Roots
$x = \pm 3$	Simplify Radicals

Example: Solve $(2x – 5)^2 + 5 = 3$

$(2x - 5)^2 + 5 = 3$	Given
$(2x - 5)^2 = -2$	Addition Property of Equality used to add –5 to both sides
$\sqrt{(2x - 5)^2} = \pm\sqrt{(-2)}$	Extract Square Roots
$2x - 5 = \pm i\sqrt{2}$	Simplify Radicals and Apply Definition of "i"
$2x = 5 \pm i\sqrt{2}$	Addition Property of Equality
$x = (5 \pm i\sqrt{2}) / 2$	Division Property of Equality

Solving Quadratic Equations – Method 3 – Completing The Square

This method of solving quadratic equations is straightforward, but requires a specific sequence of steps. Here is the procedure:

Example: Solve $3x^2 + 4x – 7 = 0$ By Completing The Square

1. **Isolate the x^2 and x-terms on one side of = by applying the Addition Property of Equality.**
   $3x^2 + 4x = 7$

2. **Apply the Division Property of Equality to divide all terms on both sides by the coefficient on x^2.**
   $(3x^2)/3 + (4x)/3 = 7/3$
   $x^2 + (4/3)x = 7/3$ *Note: Steps 1 and 2 may be done in either order.*

3. **Take ½ of the coefficient on x. Square this product. Add this square to both sides using the Addition Property of Equality. In this case, we take ½ of 4/3 which is (1/2)•(4/3) = 4/6. Square 4/6 to get (4/6) •(4/6) = 16/36 = 4/9 when reduced. Add 4/9 to both sides to get**
   $x^2 + (4/3)x + 4/9 = 7/3 + 4/9$
   $x^2 + (4/3)x + 4/9 = 21/9 + 4/9$ **multiply 7/3 by 3/3 to get common denominator**
   $x^2 + (4/3)x + 4/9 = 25/9$ **add fractions**

4. **Factor the left side.**
   Note: It will *always* factor as (x ± the square root of what you added)^2
   $(x + 2/3)^2 = 25/9$

5. **Solve by extracting square roots.**
   $\sqrt{(x + 2/3)^2} = \pm\sqrt{25/9}$ **Extract Square Roots**
   $x + 2/3 = \pm 5/3$ **Simplify Radicals**
   $x = -2/3 \pm 5/3$ **Addition Property of Equality**

This results in two answers: $x = -2/3 + 5/3 = 3/3 = 1$ *and* $x = -2/3 - 5/3 = -7/3$

You may have noticed that we solved this same problem earlier in a much easier fashion by factoring! So why learn this method of extracting square roots? Answer: This method is used in higher levels of math (like calculus) to perform similar or identical equation rearrangements. Also, we need this method to justify and derive the *Quadratic Formula*.

Solving Quadratic Equations – Method 4 – Using The Quadratic Formula
Solving a quadratic equation that is in the form $ax^2 + bx + c = 0$ only involves plugging a, b, and c into the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve $(x + 3)^2 = x - 2$

$(x + 3)^2 = x - 2$	**Given**
$(x + 3)(x + 3) = x - 2$	**Rewrite**
$x^2 + 6x + 9 = x - 2$	**Multiply out with Distributive Property, Combine Like Terms**
$x^2 + 5x + 11 = 0$	**Addition Property of Equality - add 2, add –x to both sides**

Plug a=1, b=5, c =11 from $1x^2 + 5x + 11 = 0$ into the Quadratic Formula to get

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot (1) \cdot (11)}}{2\,(1)}$$

which simplifies to

$$x = \frac{-5 \pm \sqrt{19}\; i}{2}$$

after we simplify the radical and rewrite $\sqrt{-19}$ as $(\sqrt{19}) \bullet i$ by applying the definition of i.

Where does this formula come from? Answer: Solve $ax^2 + bx + c = 0$ by completing the square! The answer, after simplification, will match the Quadratic Formula and it will be in terms of the coefficients a, b, and c. This is left as an exercise.